Turbulence and chaos:

Turbulence is $\approx$ non-repeating motion with structure across a wide range of spatial scales.

How do we develop small-scale structures with the same boundary conditions?

Typically, we see turbulence develop as the speed of the flow is increased.

N. S. equation

$$\rho \left[\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right] \vec{v} = - \vec{\nabla} p + \mu \nabla^2 \vec{V}$$

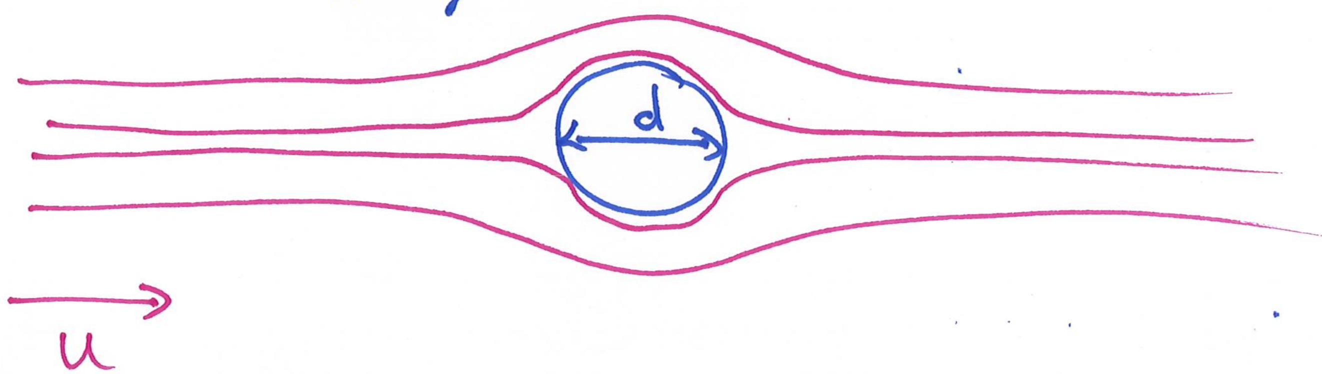
What are the dimensionless numbers in N.S. that characterize the system?

Assume incompressible flow $\rho = \rho_0$ = constant.

$\Rightarrow$ Choose a representative length and velocity scale for the system?

length scale : $L_0 = d$

velocity scale : $V_0 = u$



Redefine our variables in terms of these quantities.

$$\vec{V} = V_0 \vec{V}'$$

$\vec{V}'$ $\rightarrow$ dimensionless has all the <u>shape</u> information

V_0: constant ~ 2 m/s

$$\vec{\nabla} = \frac{1}{L_0} \vec{\nabla}'$$

$\vec{\nabla}'$ $\rightarrow$ dimensionless

$$x = L_0 x'$$

x' $\rightarrow$ dimensionless just a number.

What about t?

$$t_0 = \frac{d}{u} = \frac{L_0}{V_0}$$

$$t = t_0 \, t' \longrightarrow \text{dimensionless}$$

$$= \frac{L_0}{V_0}$$

What about pressure?

$$p = \rho_0 V_s^2 \qquad V_s = c \, V_s' \longrightarrow \text{dimens.}$$

c → representative sound speed.

$$p = (\rho_0 c^2) V_s'^2$$

$$= (\rho_0 c^2)\, p' \longrightarrow \text{dimensionless pressure.}$$

⇒ Plug into N.S.

$$\rho_0\left[\frac{1}{\cancel{t_0}}\frac{\partial}{\partial t'} + \underbrace{(V_0 \vec{v}')\cdot\left(\frac{1}{L_0}\vec{\nabla}'\right)}_{\frac{V_0}{L_0}(\vec{v}'\cdot\vec{\nabla}')}\right] V_0 \vec{v}'$$

∂t

$\frac{V_0}{L_0}\frac{\partial}{\partial t'}$

$$= -\left(\frac{1}{L_0}\vec{\nabla}'\right)(\rho_0 c^2)\, p' + \mu\left(\frac{1}{L_0^2}\nabla'^2\right)V_0\vec{v}$$

LHS:
$$\underbrace{\rho_0 \frac{V_0^2}{L_0}}_{\substack{\text{dimension}\\ \text{actual units.}}} \underbrace{\left[\frac{\partial}{\partial t'} + (\vec{v}'\cdot\vec{\nabla}')\right]\vec{v}'}_{\substack{\text{dimensionless}\\ \text{shape only}}}$$

RHS:
$$-\underbrace{\left(\frac{\rho_0 c^2}{L_0}\right)}_{\text{dimension}} \underbrace{\vec{\nabla}' p'}_{\text{dimensionless}} + \underbrace{\left(\frac{\mu V_0}{L_0^2}\right)}_{\text{dimension}} \underbrace{\nabla'^2 \vec{v}'}_{\text{dimensionless}}$$

Divide everything by $\frac{\rho_0 V_0^2}{L_0}$

$$\frac{\dfrac{\cancel{\rho_0}\, c^2}{\cancel{L_0}}}{\dfrac{\cancel{\rho_0}\, V_0^2}{\cancel{L_0}}} = \frac{c^2}{V_0^2} = \frac{1}{M^2}$$

Mach number: $M = \dfrac{V}{c}$

kinematic viscosity

$$\frac{\dfrac{\mu V_0}{L_0^2}}{\dfrac{\rho_0 V_0^2}{L_0}} = \frac{\mu_0/\rho}{V_0 L_0} = \frac{\eta}{V_0 L_0}$$

Reynolds number: $Re = \dfrac{V_0 L_0}{\eta} = \dfrac{\text{inertia}}{\text{dissipation}}$

heavy stone : lots of inertia

→ not affected by drag

ping pong ball : small inertia

→ quickly reaches terminal velocity

Dimensionless form of N.S.

$$\left[\frac{\partial}{\partial t'} + \vec{V}'\cdot\vec{\nabla}'\right]\vec{V}' = \boxed{-\frac{1}{M^2}}\vec{\nabla}'p' + \overbrace{\boxed{\frac{1}{Re}}\nabla'^2\vec{V}'}^{\text{dissipation}}$$

The entire system (for a given shape of boundary conditions) is characterized by

M and Re.

Connection to algebra:

$$x = \boxed{3}x^2 - \boxed{2}$$

the solution (roots) depend on the specific numerical values of the coefficients.

$$3x^2 - x - 2 \rightarrow x = \frac{1 \pm \sqrt{1+24}}{6}$$

$$x = \frac{1 \pm 5}{6} = \begin{cases} 1 \\ -\frac{2}{3} \end{cases}$$

Why turbulence?

if Re is large then dissipation is small.

high speed = large Re = low dissipation

allows small scale structures to develop & grow before they decay by friction.

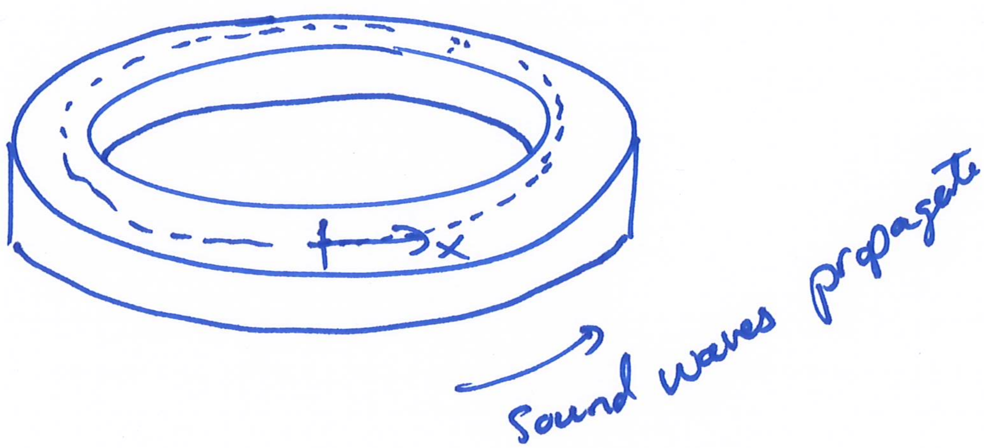
This explains the condition necessary to support small scale structure, but doesn't explain why.

~~Where do~~

From where do small scales come?

At high flow (large Re) we ignore viscous effects.

$$\left[\frac{\partial}{\partial t'} + \vec{v}'\cdot\vec{\nabla}'\right]\vec{v}' = -\frac{1}{M^2}\vec{\nabla}' p'$$



Guess a solution of the form:

$$v(x,t) = A\sin(kx-\omega t)$$

~~Extra work to show~~ $\cancel{\vec{p} \propto \frac{d}{dx} v}$

$$\cancel{\nabla^2 p = \frac{d}{dx} p = \frac{d}{dx}\left(\frac{d}{dx} v\right)}$$

$$\cancel{= k^2 v'}$$

if we had

$$\frac{\partial}{\partial t} \vec{v}' = -\frac{1}{M^2} \vec{\nabla}' p$$

then you get the wave equation

let $v = \frac{d}{dt}(d)$ displacement of air.

$$p \propto \frac{d}{dx} d$$

$$\hookrightarrow \frac{\partial^2}{\partial t^2} d = \frac{1}{v^2} \frac{\partial^2}{\partial x^2} d$$

$$\downarrow$$

$$d = A \sin(kx - wt)$$

What happened to $(\vec{v}' \cdot \vec{\nabla}') \vec{v}'$?

$$\vec{v}' = A \sin(kx - wt) (\hat{x})$$

$$\boxed{(\vec{V}\cdot\vec{\nabla}')\vec{V}'} = A\sin(kx-\omega t)\frac{\partial}{\partial x}A\sin(kx-\omega t)$$

$$= A^2 k \sin(kx-\omega t)\cos(kx-\omega t)$$

$\Rightarrow$ We ignore when A is small.

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

$$= \frac{1}{2}A^2 k \sin\left((2k)x - (2\omega)t\right)$$

$\Rightarrow$ doubles the frequency and halves the wavelength.

The $(\vec{V}\cdot\vec{\nabla})\vec{V}$ term is the primary source of turbulence generation.