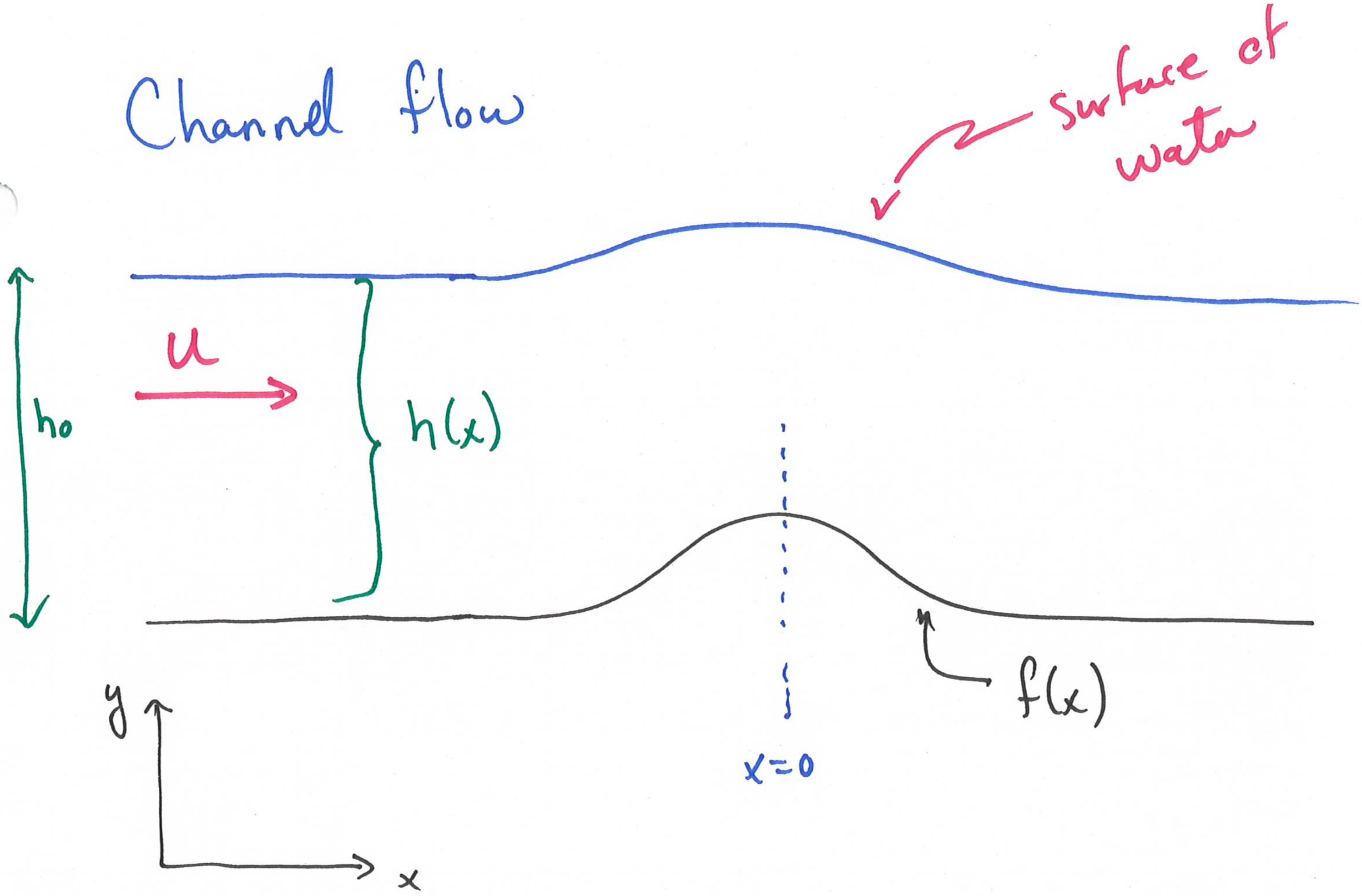
Channel flow



Q: How does the flow respond to the bump?

Our assumptions: incompressible flow $\vec{\nabla}\cdot\vec{v}=0$

$\rho = \text{constant}$

our equations:

(1) constant mass flux $\rho\, \boxed{v(x)\, h(x)} = \underbrace{\rho\, u\, h_0}_{\text{constant}}$

want to solve for $v(x)$ and $h(x)$.

② Bernoulli equation:

upstream downstream

$$p_0 + \frac{1}{2}\rho U^2 + \rho g h_0 = p + \frac{1}{2}\rho v^2 + \rho g (h + f)$$

$p = p_0$

measure the amount of mass

mass is lifted up.

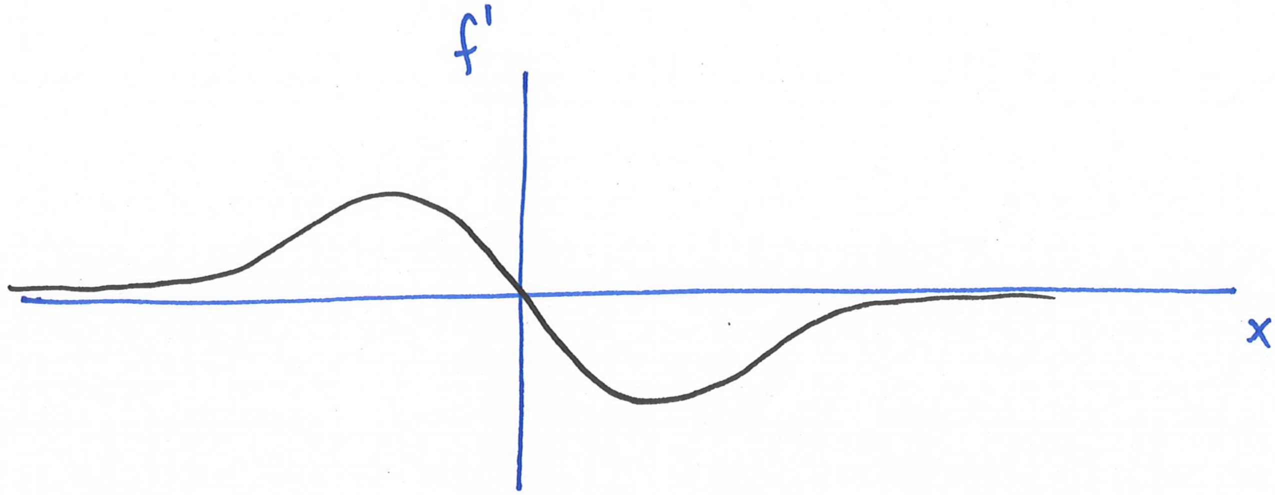
⇒ want to merge these equations.

⇒ want to develop a differential equation that relates these quantities

⇒ take a derivative with the respect to x:

① $\frac{d}{dx}(vh) = v h' + v' h = 0$

② $\frac{d}{dx}\left(\frac{1}{2}v^2 + g(h+f)\right) = v v' + g(h' + f') = 0$

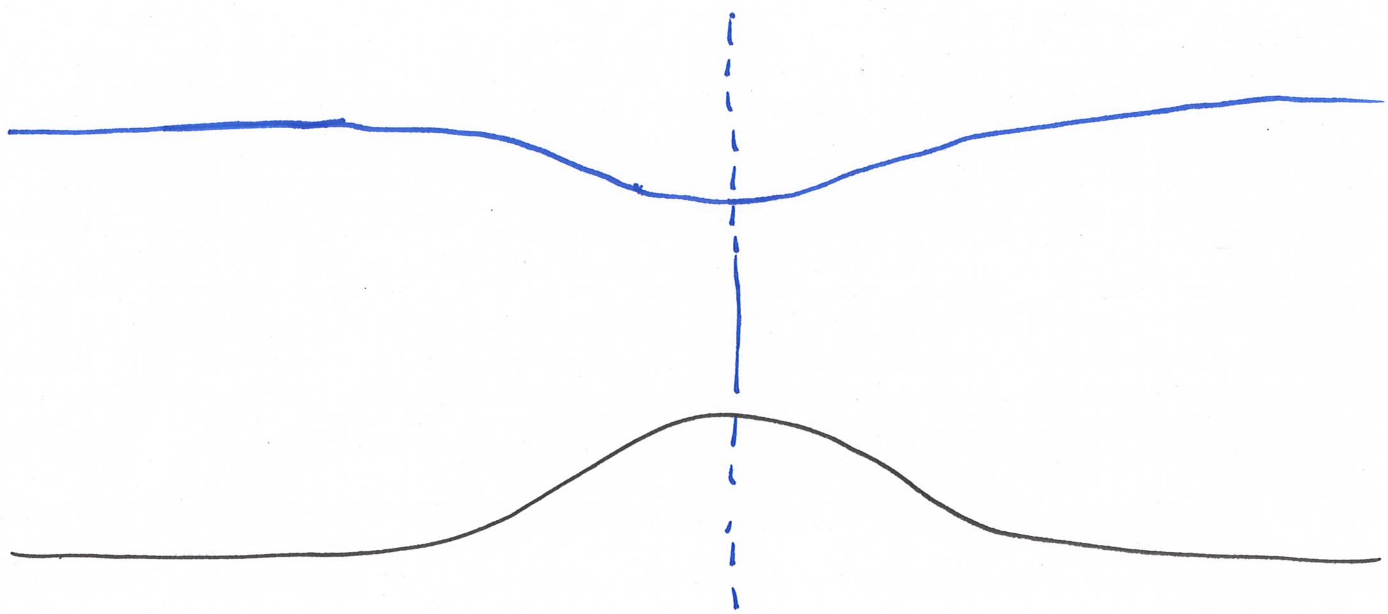


<u>case i</u> $f'=0 \Rightarrow V'=0$

$$\Downarrow$$

$h'=0$

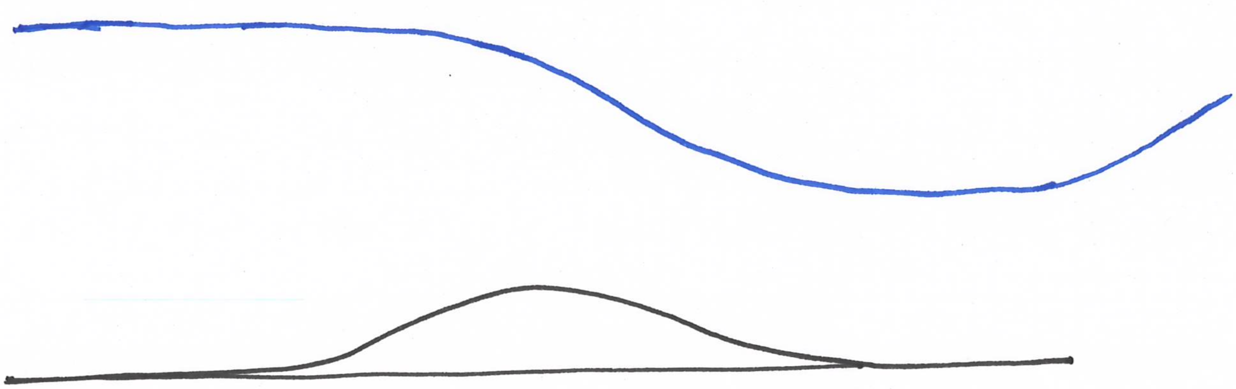
(mass conservation)



Case ii $\quad gh - v^2 = 0$

$\Rightarrow$ not true that $\frac{dv}{dx} = 0$

$\frac{dv}{dx}$ can stay the same sign



Flow becomes asymmetric and develops reduced height $\Rightarrow$ accelerated flow

$$g f' = gh\left(1 - \underbrace{\frac{v^2}{gh}}_{Fr\ =\ \text{Froude }\#}\right)\frac{v'}{v}$$

Behavior changes as Fr passes through 1
$\Rightarrow$ changes sign of the DE.

solving ① for h' :

$$h' = -h \frac{v'}{v}$$

insert into ② :

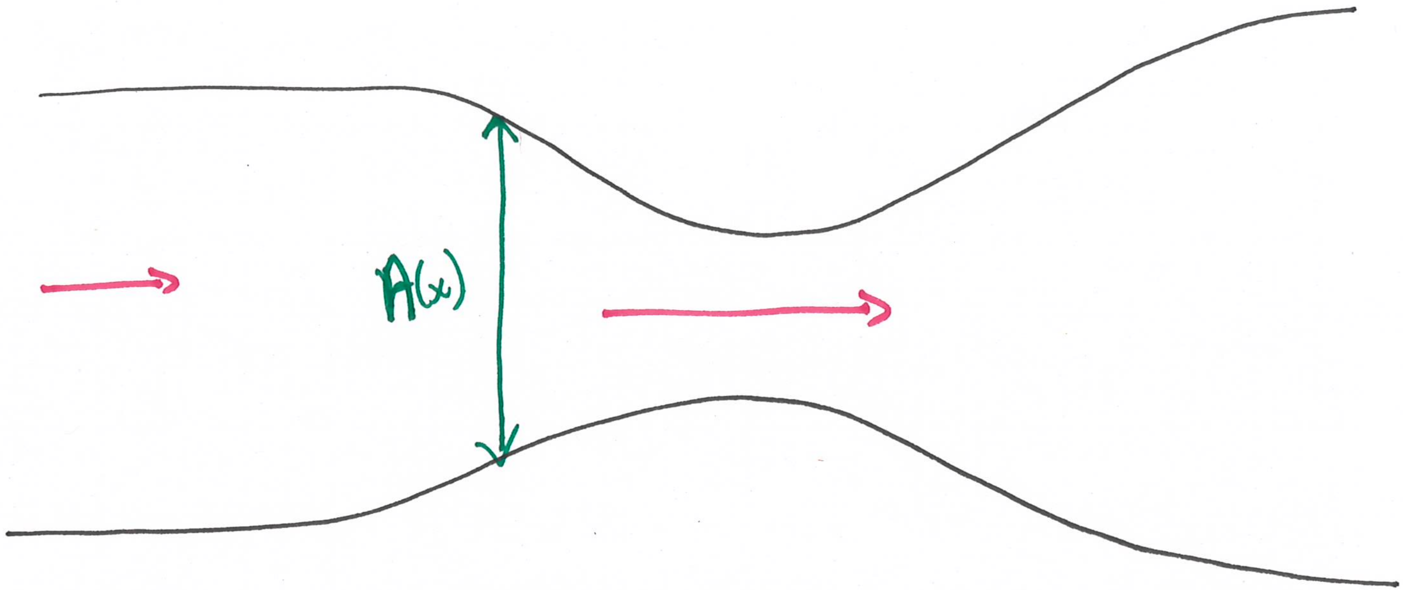
$$\underbrace{v v' + g\left(-h \frac{v'}{v}\right)}_{\text{move to the other side}} + g f' = 0$$

$$g f' = g h \frac{v'}{v} - v v'$$

$$\boxed{g f' = \left(g h - v^2\right) \frac{v'}{v}}$$

Q: What happens at the point where $f' = 0$?

Supersonic flow:



From analysis of Venturi tube, we know flow will accelerate in the neck.

Q: What happens to density at high speeds?

① conservation of mass

$$\rho(x)\, v(x)\, A(x) = \text{constant}.$$

$$\rho' v A + \rho v' A + \rho v A' = 0$$

-> divide by $\rho v A$

$$\boxed{\frac{\rho'}{\rho} + \frac{v'}{v} + \frac{A'}{A} = 0}$$

~~② Bernoulli equation:~~

$$\cancel{p_0 + \frac{1}{2}\rho u^2 = p(x) + \frac{1}{2}\rho(x)\, v(x)^2}$$

② Navier - Stokes: $\frac{\partial}{\partial t} = 0$ (steady flow)

ignore viscosity

only flow is in x-direction

$$\rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla} p \Rightarrow \rho v \frac{d}{dx} v = -\frac{d}{dx} p$$

$$\boxed{\rho v v' = -p'}$$

③ How do we connect p and ρ?

$\Rightarrow$ need to define our thermodynamic process for compression.

adiabatic compression $\Rightarrow$ fast

$\Rightarrow$ no time for diffusion

$\Rightarrow$ no heat transfer

Adiabatic equation:

$$p V^\gamma = \text{constant}$$

volume

$$\rho \sim \frac{1}{V} \Rightarrow p \rho^{-\gamma} = \text{constant}.$$

$\Rightarrow$ take a derivative:

$$p' \rho^{-\gamma} + p(-\gamma) \rho^{-\gamma-1} = 0$$

③ $$\boxed{\frac{p'}{p} = \gamma \frac{\rho'}{\rho}}$$

② $$\boxed{\rho v v' = -p'}$$

① $$\boxed{\frac{\rho'}{\rho} + \frac{v'}{v} + \frac{A'}{A} = 0}$$

$\Rightarrow$ want an equation that relates v and A.

$\Rightarrow$ eliminate p and ρ.

$\cancel{(1) + 2\,(2)}$

$(2) \longrightarrow (3)$

$$-\frac{\rho V V'}{P} = \gamma \frac{\rho'}{\rho}$$

$$\frac{\rho'}{\rho} = -\frac{\rho}{\gamma P} V V'$$

$\Rightarrow$ put into (1)

$$\underbrace{-\frac{\rho}{\gamma P} V V' + \frac{V'}{V}}_{\text{move to other side}} + \frac{A'}{A} = 0$$

$$\frac{A'}{A} = \left(\frac{\rho}{\gamma P} V^2 - 1 \right) \frac{V'}{V}$$

$\frac{\rho}{\gamma P} \rightarrow \frac{1}{V_s^2}$ $\qquad M = \frac{V}{V_s}$

$$\boxed{\frac{A'}{A} = (M^2 - 1) \frac{V'}{V}}$$

you want v to increase.

$$v' > 0$$

$$V = -v \frac{A'}{A} > 0$$

$$\Rightarrow A' < 0$$

to accelerate flow, you constrict it. when $M < 1$

eventually $v = v_s$ $(M = 1)$

for $M > 1$ to icrease flow, you need $A' > 0$