Lagrange Points Part III

Calculating the radial derivatives:

$$U(r) = -\frac{G M_E m}{r_a} - \frac{G M_s m}{r_b} + \frac{1}{2}\frac{l^2}{m r^2}$$

$$l = m r^2 \dot{\theta}$$

→ look for points r that are stable and rotate at w_0

$$U(r,\theta) = \underbrace{-\frac{G M_s m}{a}}_{V_0}\left[\underbrace{\frac{M_E}{M_s}}_{\alpha}\frac{a}{r_a} + \frac{a}{r_b}\right] + \frac{1}{2}\frac{l^2}{m r^2}$$

$$\frac{\partial V}{\partial r} = V_0\left[\alpha\frac{a}{r_a^2}(-1)\frac{\partial r_a}{\partial r} + \frac{a}{r_b^2}(-1)\frac{\partial r_b}{\partial r}\right] - \frac{l^2}{m r^3}$$

$$\frac{\partial r_a}{\partial r} = \frac{r - a\cos(\theta)}{(r^2 + a^2 - 2ar\cos(\theta))^{1/2}} = \frac{r - a\cos(\theta)}{r_a}$$

$$\frac{\partial r_b}{\partial r} = \frac{r + b\cos(\theta)}{(r^2 + b^2 + 2rb\cos(\theta))^{1/2}} = \frac{r + b\cos(\theta)}{r_b}$$

$$\frac{\partial V}{\partial r} = -V_0\left[\frac{\alpha a(r-a\cos(\theta))}{r_a^3} + \frac{a(r+b\cos(\theta))}{r_b^3}\right] - \underbrace{\frac{(mr^2\omega_0)^2}{mr^3}}_{-mr\omega_0^2}$$

$$\omega_0^2 = \frac{GM_s}{a^3(1+\alpha)^2} = \frac{-V_0}{m a^2(1+\alpha)^2}$$

$$= -V_0\left[\frac{\alpha a(r-a\cos(\theta))}{r_a^3} + \frac{a(r+b\cos(\theta))}{r_b^3} - \frac{r}{a^2(1+\alpha)^2}\right]$$

Now we need to evaluate cases:

<u>case i</u> $\quad r_a = r_b = r_0$

set $\frac{\partial V}{\partial r} = 0$

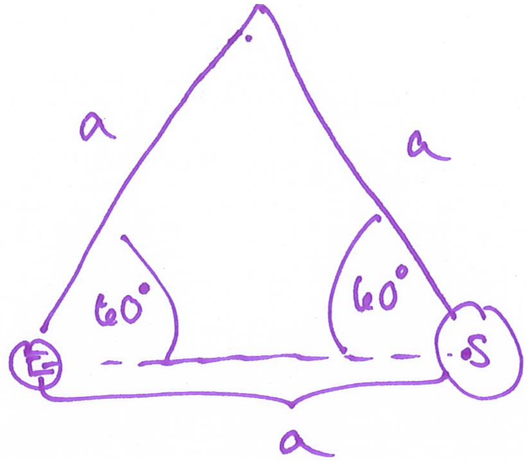
$$\frac{\alpha a r}{r_0^3} + \frac{a r}{r_0^3} + \frac{a}{r_0^3}\left[\cancel{b-\alpha a}\right]^{\to 0}\cos(\theta) - \frac{r}{a^2(1+\alpha)^2} = 0$$

$$(1+\alpha)\frac{a\cancel{r}}{r_0^3} - \frac{\cancel{r}}{a^2(1+\alpha)^2} = 0$$

$$\Rightarrow \boxed{r_0^3 = a^3(1+\alpha)^3}$$

For Earth-Sun $a \simeq 1.0$

$r_0 \simeq a$



<u>case ii</u> $\theta = 180^\circ$ $\cos(180^\circ) = -1$

$$r - a\cos(\theta) \longrightarrow r + a$$

$$r + b\cos(\theta) \longrightarrow r - b$$

$$r_a = \sqrt{r^2 + a^2 - 2ar\cos(\theta)} = r + a$$

$$r_b = \sqrt{r^2 + b^2 + 2ab\cos(\theta)}$$

$$= \sqrt{(r-b)^2} = |r - b|$$

$$= \begin{cases} r - b & \text{if } r > b \\ b - r & \text{if } r < b \end{cases}$$

does not exist.

~ 10^{-6}
ignore

$$\frac{\alpha a \cancel{(r+a)}}{(r+a)^{\cancel{3}\,2}} + \frac{a\cancel{(r-b)}}{(r-b)^{\cancel{3}\,2}} - \frac{r}{a^2(1+\alpha)^2} = 0$$

$$\frac{a}{(r-b)^2} = \frac{r}{a^2(1)^2} \Rightarrow \frac{a}{r^2} = \frac{r}{a^2}$$

$$\boxed{r = a}$$

$b = \alpha a$
ignore.