Lagrange Points: Part II

Last time we found

$$\boxed{\omega_0^2 = \frac{G M_s}{a^3 (1+\alpha)^2}}$$

Does this expression make sense?

$$\alpha = \frac{M_E}{M_S} << 1$$

What should we expect for a rotational frequency?

Guess $\boxed{\omega_0^2 = \frac{G M_{tot}}{r_{tot}^3}}$

$$M_{tot} = M_S + M_E = M_s\left(1 + \frac{M_E}{M_s}\right)$$

$$= M_s(1+\alpha)$$

$$r_{tot} = a + b = a + \alpha a = a(1+\alpha)$$

$$\omega_0^2 = \frac{G M_s \cancel{(1+\alpha)}}{a^3 (1+\alpha)^{\cancel{3}}} = \frac{G M_s}{a^3 (1+\alpha)^2}$$

Returning to the potential:

$$V(r,\theta) = -\frac{G M_s m}{r_b} - \frac{G M_E m}{r_a} + \frac{1}{2} m r^2 \omega_0^{\,2}$$

pull out a factor of $-\frac{G M_s m}{r} = V_0$

$$V(r,\theta) = -\frac{G M_s m}{r}\left[\frac{r}{r_b} + \frac{M_E}{M_s}\frac{r}{r_a} - \frac{1}{2}\frac{r^3 \omega_0^2}{G M_s}\right]$$

$$= V_0\left[\frac{r}{r_b} + \alpha\frac{r}{r_a} - \frac{1}{2}\frac{r^3}{a^3(1+\alpha)^2}\right]$$

To find stability, we need two things.

$$\frac{\partial V}{\partial r} = 0 \qquad \text{and} \qquad \frac{\partial V}{\partial \theta} = 0$$

Easier to calculate $\frac{\partial V}{\partial \theta}$ first.

$$\frac{\partial V}{\partial \theta} = V_0 \left[\left(\frac{-r}{r_b^2} \right) \frac{dr_b}{d\theta} + \alpha \left(\frac{-r}{r_a^2} \right) \frac{dr_a}{d\theta} \right]$$

$$r_b = \sqrt{r^2 + b^2 + 2rb\cos(\theta)}$$

$$= \left(r^2 + b^2 + 2r.b\cos(\theta) \right)^{1/2}$$

$$\frac{dr_b}{d\theta} = \frac{1}{\cancel{2}} \underbrace{\left(r^2 + b^2 + 2rb\cos(\theta) \right)^{-1/2}}_{\frac{\ }{r_b}} \left(-\cancel{2} rb\sin(\theta) \right)$$

$$= \frac{-rb\sin(\theta)}{r_b}$$

$$\frac{dr_a}{d\theta} = + \frac{ra\sin(\theta)}{r_a}$$

$$\frac{\partial V}{\partial \theta} = -V_0 \left[\frac{r}{r_b^2} \left(-\frac{r \cdot b\sin(\theta)}{r_b} \right) + \alpha \frac{r}{r_a^2} \left(\frac{r \cdot a\sin(\theta)}{r_a} \right) \right]$$

$$= -V_0 r^2 \sin(\theta) \left[-\frac{b}{r_b^3} + \alpha \frac{a}{r_a^3} \right]$$

Want $\frac{\partial V}{\partial \theta} = 0$.

need $\sin(\theta) = 0$

$$\theta = n\pi$$

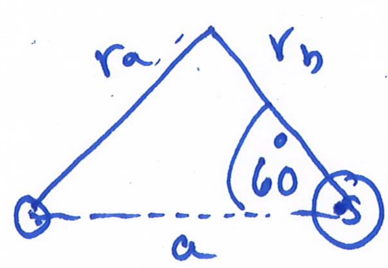
$$\theta = \begin{cases} 0 \\ \pi \end{cases}$$

or $-\frac{b}{r_b^3} + \alpha \frac{a}{r_a^3} = 0$

recall $b = \alpha a$

$$\Rightarrow r_b = r_a$$

For a system with $\alpha \ll 1$



$r_a \approx r_b \approx a$

$$\boxed{\theta = 0, \pm 60°, 180°}$$

and also possibly $r = 0$