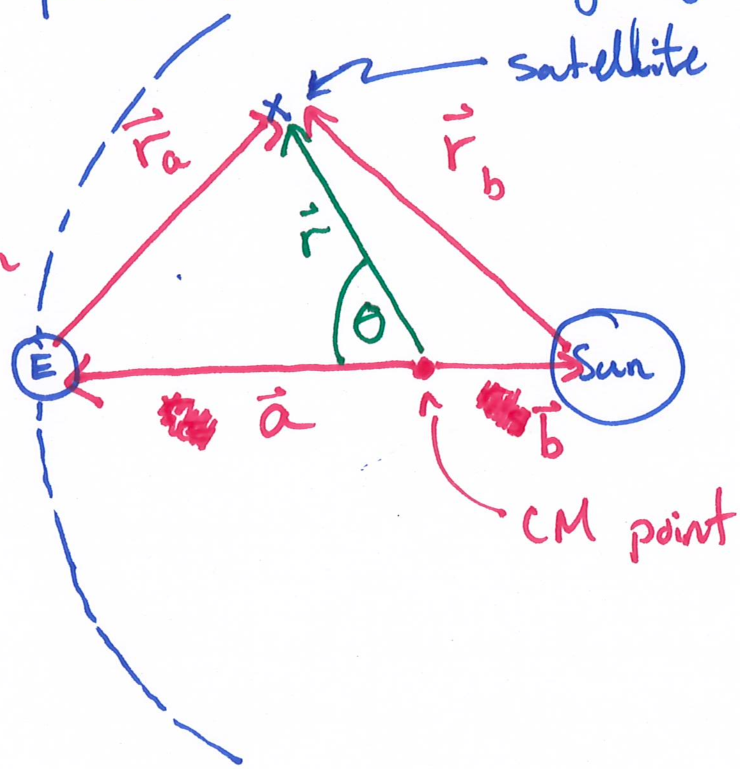
Lagrange Points: Part 1

- stable points in a 3-body system:



Masses of our bodies

$m_{sat} << m_E << m_{sun}$

A stable point means that ....

① zero net force at that point

② configuration (when viewed in a rotating reference frame) does not change.

↳ Around what point does the system rotate?

Strategy: ① Calculate effective potential

$V(r, \theta)$

includes centrifugal force

(in rotating frame)

② take derivatives to solve for $\vec{F}$,

③ set $\vec{F} = 0$

defines stability.

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① Effective potential:

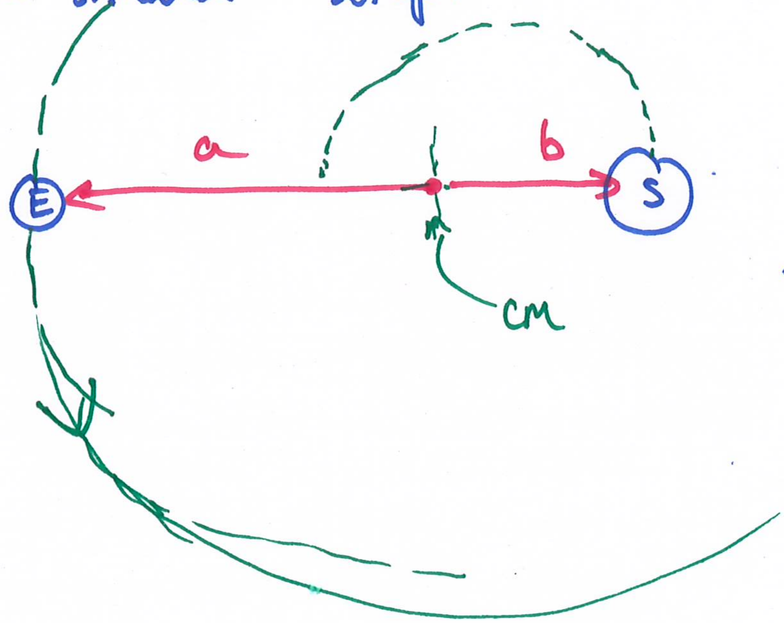
$$V(r,\theta) = -\frac{G M_S m}{r_b} - \frac{G M_E m}{r_a} + \frac{1}{2} m r^2 \omega_0^2$$

$\dot{\theta}$

What is ω for this system?

$\omega_0 = \omega_{\text{Earth-Sun system}}$

To calculate w_0, we can use a simalar setup.



Relationship between a & b:

$$M_E a = M_s b$$

$$b = \frac{M_E}{M_s} a = \alpha a$$

$$\alpha = \frac{M_E}{M_s} \ll 1 \quad (\text{small \#})$$

Force analysis for the Earth:

radial force balance

$$F_r = \frac{-G M_s M_E}{(a+b)^2} + M_E w_0^2 a = 0$$

$$\omega_0^2 = \frac{G M_s}{a(a+b)^2} \qquad b = \alpha a$$

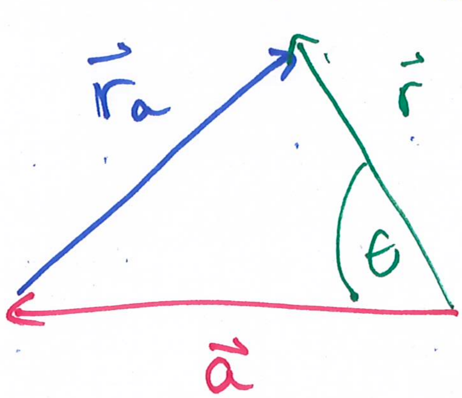
$$\boxed{\omega_0^2 = \frac{G M_s}{a^3(1+\alpha)^2}}$$

To calculate stability, we need the forces. $\Rightarrow$ take derivatives of the potential.

$$\frac{\partial V}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial V}{\partial r} = 0$$

Need $r_a = r_a(r, \theta)$

$r_b = r_b(r, \theta)$



$$\vec{r} = \vec{a} + \vec{r}_a$$

$$\vec{r}_a = \vec{r} - \vec{a}$$

Law of cosines:

$$\vec{r}_a \cdot \vec{r}_a = r_a^2$$

$$(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{a}) = \vec{r} \cdot \vec{r} - \vec{r} \cdot \vec{a}$$
$$- \vec{a} \cdot \vec{r} + \vec{a} \cdot \vec{a}$$

$$\vec{a} \cdot \vec{r} = \vec{r} \cdot \vec{a}$$

$$\vec{a} \times \vec{r} = - \vec{r} \times \vec{a}$$

$$= r^2 - 2 \vec{r} \cdot \vec{a} + a^2$$

$$= r^2 + a^2 - 2 r a \cos(\theta)$$

$$r_a = \left(r^2 + a^2 - 2 r a \cos(\theta) \right)^{1/2}$$

$$r_b = \left(r^2 + b^2 + 2 r b \cos(\theta) \right)^{1/2}$$

$$= \left(r^2 + b^2 - 2 r b \underbrace{\cos(180 - \theta)}_{-\cos(\theta)} \right)^{1/2}$$