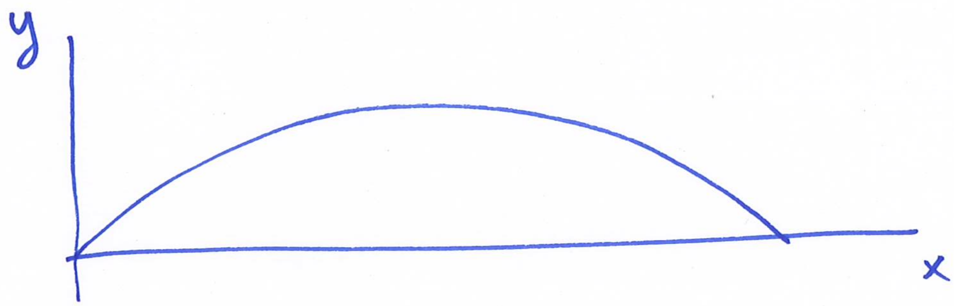
What is meant by $\frac{d}{dt}$?

An example from kinematics:

$$y(t) = y_i + V_{yi}\, t + \tfrac{1}{2} a_y t^2$$

$$x(t) = \cancelto{0}{x_i} + V_{xi}\, t$$



if we wanted V_y

$$V_y = \frac{d}{dt} y = \boxed{V_{yi} + a_y t}$$

replace some of the t's with x's

$$t = \frac{x}{V_{xi}}$$

$$y(x,t) = y_i + \frac{V_{yi}}{V_{xi}} x + \tfrac{1}{2} a_y t^2$$

Now, calculate V_y again.

$$V_y = \frac{d}{dt} y$$

$$= a_y t + \frac{\partial y}{\partial x} \frac{\partial x}{\partial t}$$

$$= a_y t + \frac{V_{y_i}}{\cancel{V_{x_i}}} \cancel{V_{x_i}}$$

$$= \boxed{a_y t + V_{y_i}}$$

⇒ look for time dependence in all of the independent variables.

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla}}$$

look in all the x, y, z

time is explicit

$$= \frac{\partial}{\partial t} + V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} + V_z \frac{\partial}{\partial z}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z}$$

Summary:

① Conservation law can be written as

$$\frac{d}{dt} f = - f(\vec{\nabla}\cdot\vec{v}) + S_f$$

② Convective derivative

$$\frac{d}{dt} f = \frac{\partial}{\partial t} f + (\vec{v}\cdot\vec{\nabla}) f$$

Conservation of momentum:

$$\text{momentum density} = \rho\,\vec{v}$$

$$\frac{d}{dt}(\rho\vec{v}) = -(\rho\vec{v})(\vec{\nabla}\cdot\vec{v}) + S_{\text{momentum}}$$

Left hand side:

$$\frac{d}{dt}(\rho\vec{v}) = \rho\frac{d}{dt}\vec{v} + \vec{v}\boxed{\frac{d}{dt}\rho}$$

$$\frac{d}{dt}\rho = -\rho(\nabla\cdot\vec{v}) + \cancel{S_{mass}}$$

$$v\frac{d}{dt}\rho = -(\rho\vec{v})(\vec{\nabla}\cdot\vec{v})$$

this cancels with first term on the RHS of mom. eqn.

$$\rho\frac{d}{dt}\vec{v} = \rho\left[\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right]\vec{v}$$

$$\rho\left[\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right]\vec{v} + \cancel{\vec{v}\frac{d}{dt}\rho}$$

$$= \cancel{-(\rho\vec{v})(\nabla\cdot\vec{v})} + S_{momentum}$$

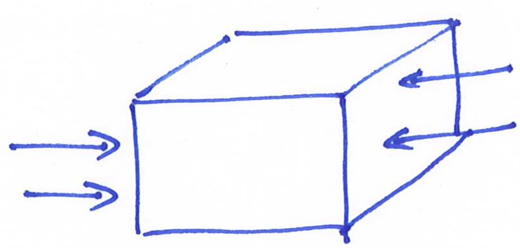
Conservation of momentum:

$$\rho\left[\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right]\vec{v} = S_{\text{momentum}}$$

What sources for momentum exist?

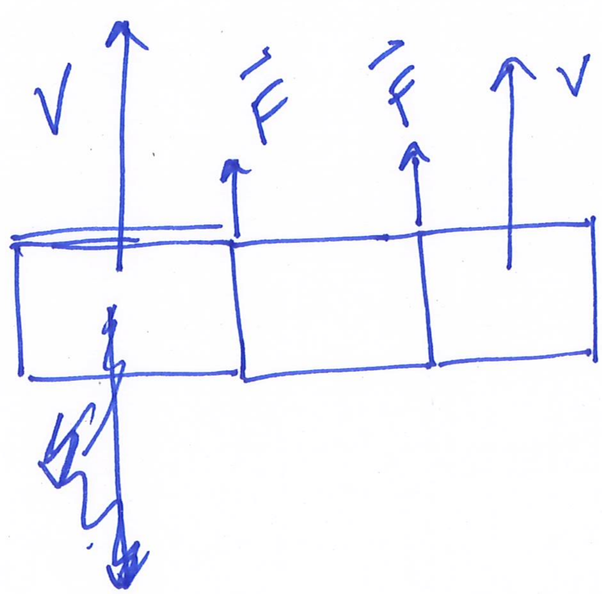
External sources: gravity (for example)

Internal sources: from microscopic world.



<u>pressure force</u>

force applied perpendicular to a surface.

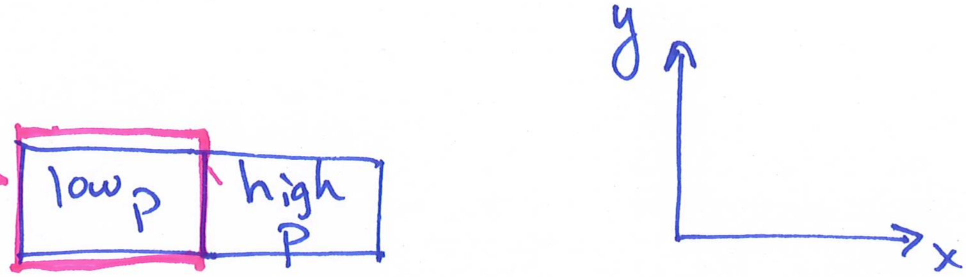


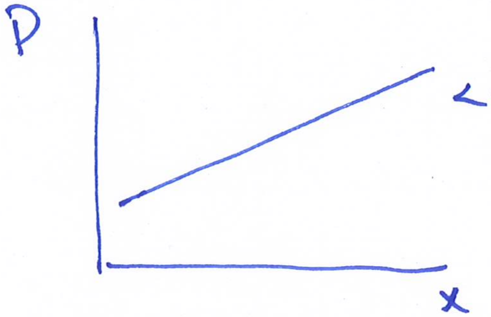
<u>viscous force</u>

force generated parallel to a surface.

Pressure force: $-\vec{\nabla} p$

What is the force on this fluid element





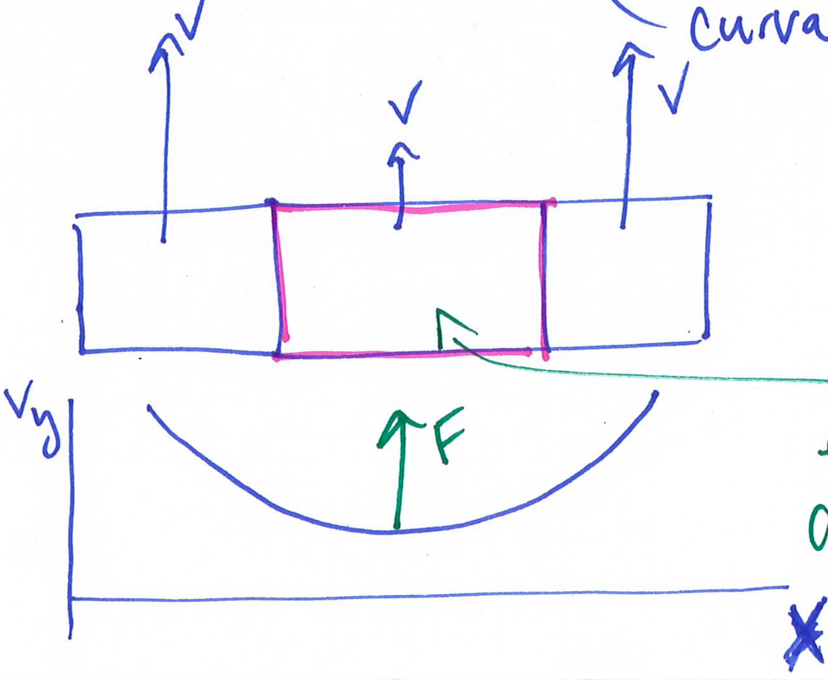
$\nabla p = \frac{d}{dx} p$

is positive

force had better be in the $-\hat{x}$ direction.

Viscous force: $\eta \, \vec{\nabla}^2 \, \vec{v}$

curvature



this fluid elements gets dragged up in speed

Navier-Stokes:

$$\boxed{\rho \underbrace{\left[\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right]\vec{v}}_{\frac{d}{dt}\vec{v}} = \vec{f}_{ext} - \vec{\nabla} p + \eta \nabla^2 \vec{v}}$$

$$\boxed{\left[\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\right]\rho = -\rho(\vec{\nabla}\cdot\vec{v})}$$