Fluid Mechanics

The governing equation for fluid mechanics is the Navier-Stokes equation:

$$\rho\left[\frac{\partial}{\partial t}\vec{v} + (\vec{v}\cdot\vec{\nabla})\,\vec{v}\right] = \rho\,\vec{f} - \vec{\nabla}p + \eta\,\vec{\nabla}^2\vec{v}$$

(scalar) ρ = fluid density

$\vec{v}$ = fluid velocity

$\vec{f}$ = sum of the external forces (force density) (e.g. gravity)

(scalar) p = fluid pressure

(scalar) η = fluid viscosity

EM

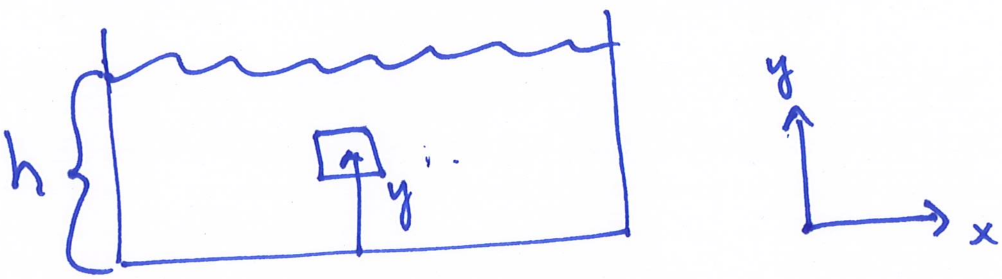
An introductory problem:

static fluid pressure $\qquad \vec{v} = 0$

N.S. equation becomes $\qquad 0 = \rho \vec{f} - \vec{\nabla} p$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$



$$\vec{f} = g(-\hat{y})$$

$$\rho\vec{f} = \rho g(-\hat{y}) \quad \leftarrow \text{like } mg(-\hat{y})$$

→ only unknown is p

uniform in the $\hat{x}$ & $\hat{z}$ direction $\quad \frac{\partial}{\partial x}, \frac{\partial}{\partial z} = 0$

$$\frac{\partial}{\partial y} p(\hat{y}) = \rho g(-\hat{y})$$

$$\frac{\partial}{\partial y} P = -\rho g$$

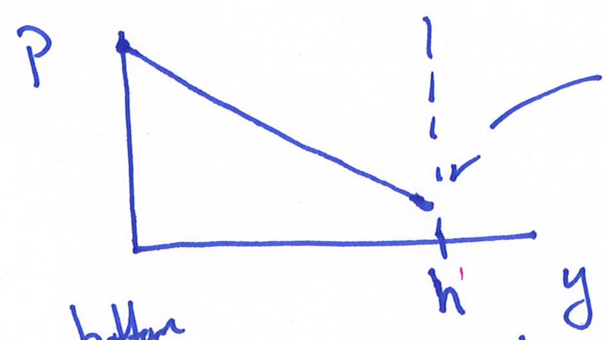
for water $\rho \simeq$ constant

$$\int_{p_i}^{p_f} \partial p = -\rho g \int_{y_i}^{y_f} \partial y$$

$$p - p_i = -\rho g (y - \cancelto{0}{y_i})$$

let $y_i = 0$, $p_i = P_{bottom}$

$$p(y) = P_{bottom} - \rho g y$$



at surface $p(h) = P_{atm}$

$$P_{atm} = P_{bottom} - \rho g h \longrightarrow P_{bottom} = P_{atm} + \rho g h$$

$$\boxed{p(y) = P_{atm} + \rho g (h - y)}$$

Guiding principles:

<u>Conservation laws</u>

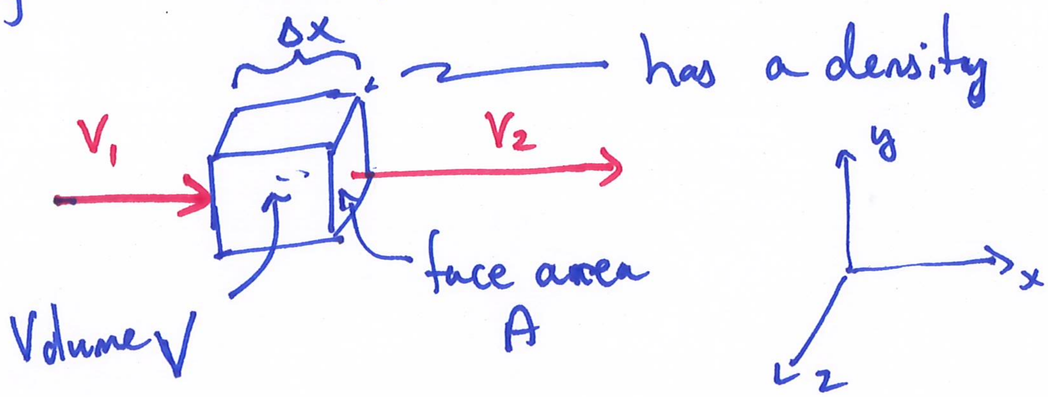
① cons. of mass

② cons. of momentum

③ cons. of energy.

A closer look at conservation of mass:

Observing a small parcel of fluid



What is the rate of change of mass in this cube of space?

$$\frac{\Delta m}{\Delta t} = \frac{(\Delta \rho)\, V}{\Delta t} \quad \text{volume of the cube}$$

= flux of mass across the boundaries

flux is measure of material transport across a surface.

$$\Gamma = \text{flux} = A \rho v \qquad V = A \Delta x$$

$$A = \frac{V}{\Delta x}$$

$$\text{flux} = \frac{V}{\Delta x} \rho v$$

gain or loss in mass is proportional to the difference in fluxes

$$\Delta m = (\Gamma_{in} - \Gamma_{out}) \Delta t$$

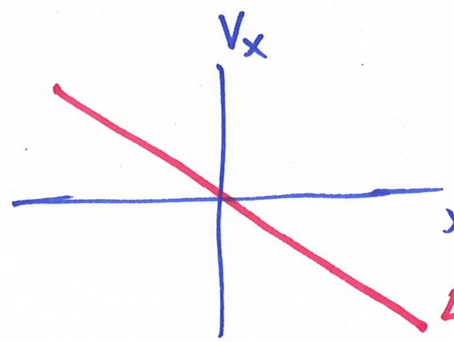
$$= \left(\frac{V}{\Delta x} \rho v_1 - \frac{V}{\Delta x} \rho v_2 \right) \Delta t$$

$$\Delta m = -\rho V \left(\frac{\overbrace{v_2 - v_1}^{\Delta v}}{\Delta x} \right) \Delta t$$

$$\frac{\Delta \rho}{\Delta t} \cancel{V} = -\rho \cancel{V} \frac{\Delta v_x}{\Delta x} \longrightarrow \boxed{\frac{d}{dt} \rho = -\rho (\nabla \cdot \vec{v})}$$

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial}{\partial x} V_x + \frac{\partial}{\partial y} V_y + \frac{\partial}{\partial z} V_z$$

For mass in-flow, $v_1 > 0 \quad v_2 < 0$

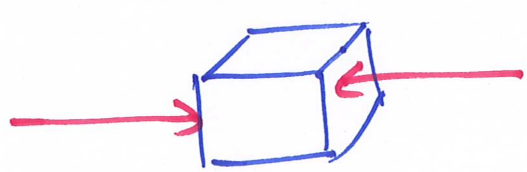


$V(x) = -\alpha x$

$$\frac{d}{dt}\rho = -\rho(\vec{\nabla} \cdot \vec{v}) = -\rho(-\alpha)$$

$$= +\rho\alpha$$

positive, which means that mass will accumulate, density will increase.



The generalized conservation law:

$$\frac{d}{dt} f = - f(\vec{\nabla} \cdot \vec{v})$$

any quantity

Another way to increase a quantity,

$$\boxed{\frac{d}{dt} f = \underbrace{- f(\nabla \cdot \vec{v})}_{\text{due to flow}} + \underbrace{S_f}_{\text{source}}}$$