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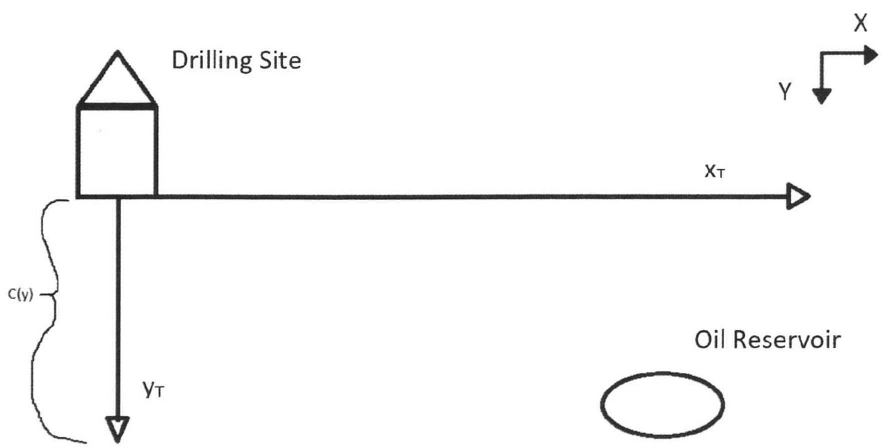
Adv. Classical Mechanics

Assignment 1 Write-up

Scenario:



Introduction:

The following analysis utilizes Lagrangian analysis and the Beltrami identity to analyze the most cost-efficient method to extract oil from an oil reservoir. The drilling site cannot sit over top of the oil reservoir, so it has been placed at a position X_T away. The soil is vertically stratified, such that digging cost increases exponentially as y increases. The vertical distance to the oil reservoir is y_T. The goal of this analysis is to find the path such that the cost per unit length at depth y, C(y) , is a minimum.

Analysis:

To begin, we must create a function which we will minimize—cost per unit length. Taking any point along the path dL, we can use Pythagorean theorem to create a function of dL which we can integrate.

$$dL = \sqrt{dx^2 + dy^2}$$

Which we can write our cost per segment as

$$f(y, y') = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, C(y)$$

In the case of exponential rate of cost increase per unit length y, we can take $C(y)=C_0e^{ay}$

Next, using the Beltrami identity, a simplified and less general version of the Euler-Lagrange equation in the calculus of variations, we can write our function as

$$f - \frac{df}{dy'}y' = k$$

Where,

$$\frac{df}{dy'} = C(y)\frac{y'}{\sqrt{1+y'^2}}$$

Thus,

$$k = C(y)\sqrt{1+y'^2} - C(y)\frac{y'^2}{\sqrt{1+y'^2}}$$

Next, we can separate items with y and x dependence to opposite sides of the equal sign, resulting in the differential equation we can evaluate

$$\frac{dy}{\sqrt{\frac{C^2}{k^2}-1}} = dx$$

We can solve by integrating

$$\int_0^{y(x)} \frac{dy}{\sqrt{\frac{C^2}{k^2}-1}} = \pm\int_0^x dx$$

With constraints $y(x_t)=y_t$, and $y(0)=0$

By observing that the LHS of the above equation takes the form

$$\int \frac{1}{\sqrt{a^2x^2-1}}dx$$

A table of integrals tells us that the result takes the form

$$\frac{1}{a}\tan^{-1}(\sqrt{a^2x^2-1}) + C$$

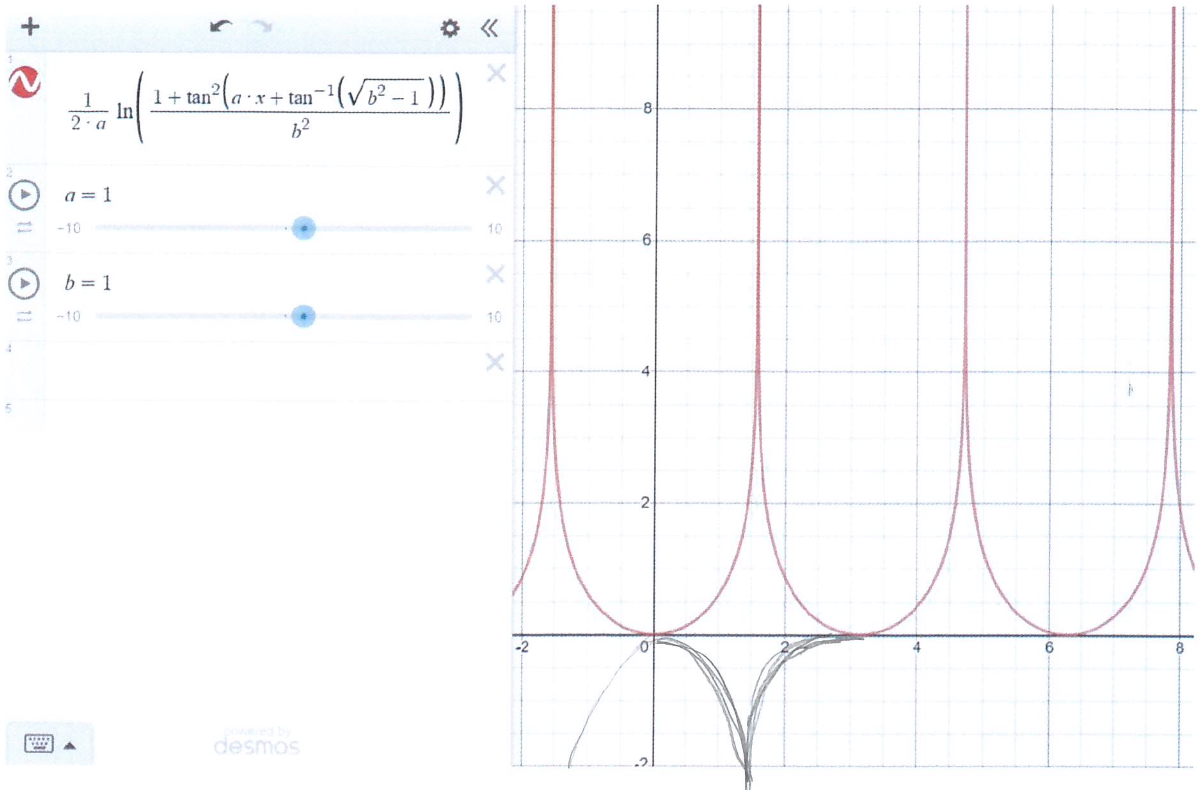
Taking C/K = b, our result from the above LHS integral is

$$\tan^{-1}(\sqrt{b^2e^{2ay}-1}) \ evaluated\ from\ y\ to\ 0 = x$$

Solving for y we get,

$$y(x) = \frac{1}{2a}\ln\left(\frac{1+\tan^2(ax+\tan^{-1}\sqrt{b^2-1})}{b^2}\right)$$

With this function, we can plug this into desmos to view our solution graphically.



This graph shows that our solution has discontinuity in it. What this means in terms of our scenario is our path of least cost consists of two segments. The first is displacing some amount x without any displacement in y, followed by an exponential path as shown below. Which is equivalent to moving our digging site closer to the oil reserve such that our second segment is only an exponential solution. Given that we must dig in the y direction before any x directional movement can occur due to the predetermined rules and regulations of this scenario, our solution only works mathematically, but physically cannot be done. Thus, we would need to have more strict definitions in our problem which may prevent this illegal move made in the first segment from happening.