1 Maximizing Projectile Range on an Inclined Plane

Scott Blankenbaker

We attempt to find the angle maximizing the range of a projectile fired with a fixed velocity on an inclined plane. The equations of motion of the projectile, and the line of the surface, are then:

$x_p(t) = x_p(0) + v_{ix} * t = v_i * cos(\theta) * t$

$y_p(t) = y_p(0) + v_{iy} * t + \frac{1}{2}a * t^2 = v_i * sin(\theta) * t - \frac{g}{2}t^2$

$y_s(x) = tan(\theta_s) * x$

Solving the first equation for t, then substituting into the second, allows us to express the projectile motion in terms of horizontal position:

$y_p(x, \theta) = tan(\theta) * x - \frac{g}{2v_i^2}x^2 sec^2(\theta)$

We then define $f(x)$ as the difference between the projectile and the ground:

$f(x, \theta) = y_p(x, \theta) - y_s(x)$

We note that the range is when this difference is equal to zero:

$f(r, \theta) = y_p(r, \theta) - y_s(r) = 0$

We then differentiate with respect to θ to begin our maximization:

$\frac{df}{d\theta} = \frac{\partial y_p}{\partial r} * \frac{dr}{d\theta} + \frac{\partial y_p}{\partial \theta} - \frac{\partial y_s}{\partial r} * \frac{dr}{d\theta} = 0$

$(tan(\theta) - \frac{g}{v_i^2} r sec^2(\theta)) * \frac{dr}{d\theta} + sec^2(\theta) * r - \frac{g}{v_i^2} r^2 sec^2(\theta) tan(\theta) - tan(\theta_s) \frac{dr}{d\theta} = 0$

$\frac{dr}{d\theta}(tan(\theta) - tan(\theta_s) - \frac{g}{v_i^2} r sec^2(\theta)) + sec^2(\theta) * r - \frac{g}{v_i^2} r^2 sec^2(\theta) tan(\theta) = 0$

$\frac{dr}{d\theta} = \frac{sec^2(\theta) * r - \frac{g}{v_i^2} r^2 sec^2(\theta) tan(\theta)}{tan(\theta) - tan(\theta_s) - \frac{g}{v_i^2} r sec^2(\theta)}$

Seeking a maximum, we set $\frac{dr}{d\theta} = 0$, noting that this requires either $\sec^2(\theta) * r - \frac{g}{v_i^2} r^2 sec^2(\theta) tan(\theta) = 0$ or $tan(\theta) - tan(\theta_s) - \frac{g}{v_i^2} r sec^2(\theta)$ tending to infinity. We note the latter occurs when $\cos(\theta) = 0$, representing shooting straight up or down; this minimizes range, and isn't our goal, so we examine the former.

$sec^2(\theta) * r - \frac{g}{v_i^2} r^2 sec^2(\theta) tan(\theta) = 0$

$r * sec^2(\theta)(1 - \frac{g}{v_i^2} r tan(\theta)) = 0$

We neglect the r = 0 case and note $sec^2(\theta)$ never equals zero, giving:

$r = \frac{v_i^2}{g * tan(\theta)}$

We return to our equations of motion, setting $y_p = y_s$, and substitute in our r.

$tan(\theta) * r - \frac{g}{2v_i^2} r^2 sec^2(\theta) = tan(\theta_s) * r$

$\frac{v_i^2}{g} - \frac{v_i^2}{2g} \frac{sec^2(\theta)}{tan^2(\theta)} = tan(\theta_s) * \frac{v_i^2}{g * tan(\theta)}$

$1 - \frac{1}{2} csc^2(\theta) = tan(\theta_s) cot(\theta)$

$sin^2(\theta) - \frac{1}{2} - tan(\theta_s) cos(\theta) sin(\theta) = 0$

$\theta = \frac{1}{2}(n * \pi - arctan(\frac{1}{tan(\theta_s)}) = \frac{1}{2}(n * \pi - (\frac{\pi}{2} - \theta_s)$

The $n * pi$ term adjusts for the repeated period of tangent over sine and cosine; we only care about the even or odd terms, so we get two solutions. The even solution gives backwards trajectories, so we choose $n = 1$, giving:

$\theta_{max} = \frac{1}{2}(\theta_s + \frac{\pi}{2})$

This is equivalent to stating that the firing angle for maximum range is halfway between the surface and vertical.

2 Supporting Figures

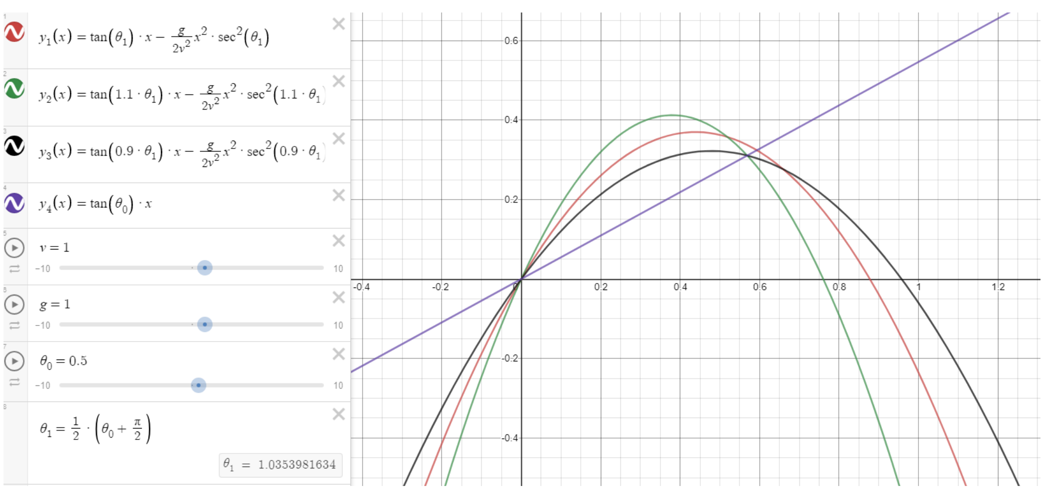

Figure 1: An upward surface (blue), with the ideal trajectory (red), and trajectories just above and below it.

Figure 2: An downward surface (blue), with the ideal trajectory (red), and trajectories just above and below it.

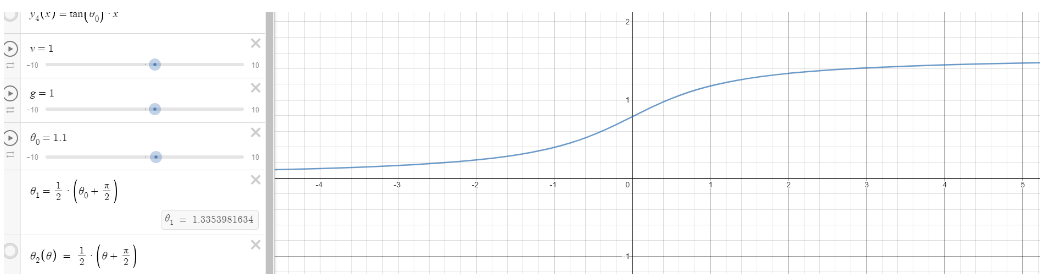

Figure 3: The relationship between ideal firing angle and slope $m = tan(\theta_s)$