

Comment on adiabatic processes and the adiabatic index

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The ubiquity of isentropic processes in heat engines and a plethora of other physical systems necessitate a discussion of the adiabatic, however, it all too often seems to be the case that derivations of the constancy of pV^γ in adiabatic processes are so unintuitive that the majority of students would be left at a loss to explain why this should be so. Dare I say that many physics faculty, myself included, are also often without a simple picture of this process. We are often forced, out of the interest of time, to accept this result as a matter of faith, a response contrary to our goals of developing a respect for critical thinking and logic that forms the foundation of our studies.

Let us consider a cylinder of length L and cross sectional area A that is filled with an ideal gas. For the purposes of this discussion we consider an ideal gas to be one in which the volume occupied by the particles is small compared to the total volume of the system and there are no long-range forces between particles so that there is no need to account for additional potential energy terms in the energy balance. Let the cylinder have a free end (a piston) that can be held fixed or allowed to move. By examining the collisions of single particles with the wall we find that the momentum transferred to the piston per unit time is

$$\Delta p = \frac{mv_x^2}{L} \Delta t$$

from which we can define the average force as

$$F = \frac{mv_x^2}{L}$$

The force per unit area is the single-particle (sp) pressure, with $V=AL$, we have

$$p_{sp} = \frac{mv_x^2}{V}$$

Considering a maxwellian distribution of the form

$$f(v_x) = \frac{1}{\sqrt{2\pi}v_{th}} \exp\left(-\frac{v_x^2}{2v_{th}^2}\right) = \frac{1}{\sqrt{2\pi}v_{th}} \exp\left(-\frac{\frac{1}{2}mv_x^2}{T}\right)$$

where we can define the temperature T in terms of the thermal velocity as

$$T = mv_{th}^2$$

Averaging the single-particle pressure over the distribution, the result is that v_x^2 is simply replaced by v_{th}^2 . Using the prior definition for the total pressure,

$$p = \frac{N}{V} T$$

An adiabatic process is one in which there is no heat flow in or out of the gas during a change of volume. One may think of such processes as happening in the limit in which the time over which a change in volume occurs to be small compared to the time scale for thermal diffusion through the boundaries. Under these conditions, we may consider the boundaries to be perfectly insulating. It follows then that any change in internal energy, the sum of particle kinetic energy plus that due to internal motions such as vibrations or rotations, can only be a function of the work done on or by the gas.

To develop a description of the adiabatic process, we will consider the case of a monoatomic gas (no internal structure in which to store kinetic energy). Let us consider, for the sake of argument, that the gas will be expanding and doing work on the piston. The argument works equally well in the reverse, but it will be easier to develop the discussion by examining a specific case. Let the gas be in thermal equilibrium with itself, that is, it is described by a single temperature, T_0 for all degrees of freedom,

$$T_{x,0} = T_{y,0} = T_{z,0} = T_0$$

If we consider an expansion of the volume due to movement of the piston through a small distance Δx , the work done by the gas is

$$W = -p dV$$

where dV is a positive number. We now need to consider the source of this work from the perspective of the motions of the gas particles. Recalling that the pressure fundamentally represents the flux of momentum through a given area, it follows that the work done by the gas is related to a change in the kinetic energy of the gas **in the direction of motion of the piston**. During this brief interval of motion of the piston, the kinetic energy of the gas in the x-direction must decrease because it is the x-motion alone that is responsible for the displacement of the piston. In the following analysis we let the subscript “0” refer to the conditions before the movement of the position, the subscript “1” refers to the intermediate state after the piston has moved but before thermal equilibrium is re-established, and the subscript “2” refers to the final state after thermal equilibrium is established.

$$W = \frac{1}{2} Nm(v_{x,1}^2 - v_{x,0}^2) = \frac{1}{2} N(T_{x,1} - T_{x,0}) = \frac{1}{2} N(T_{x,1} - T_0)$$

This expression can be inverted to solve for the intermediate x temperature prior to the re-establishment of thermal equilibrium.

$$T_{x,1} = T_0 + 2W = T_0 - \frac{2}{N} p dV$$

The y and z temperatures are unaffected by the motion of the piston,

$$T_{y,1} = T_0$$

$$T_{z,1} = T_0$$

Once the system comes back into thermal equilibrium with itself it will again have a single temperature. The equipartition of kinetic is the expression of a final temperature equal to the average of the intermediate temperatures, that is,

$$T_2 = \frac{1}{3}(T_{x,1} + T_{y,1} + T_{z,1}) = T_0 - \frac{2}{3N}pdV$$

Or, solving for the infinitesimal change in temperature, $dT=T_2 - T_0$, we have

$$dT = -\frac{2}{3}pdV$$

And using the ideal gas law to remove p from the expression, we have

$$\frac{dT}{T} = -\frac{2}{3} \frac{dV}{V}$$

from which it readily follows that

$$T = cV^{-2/3}$$

where c is a constant. This relation for temperature can be inserted into the ideal gas law,

$$pV = cNV^{-2/3}$$

For constant N, this expression can be rearranged to find

$$pV^{5/3} = cN = \text{constant}$$

The exponent of the volume is the adiabatic index, γ . For a monoatomic gas this value is $5/3$. For molecular gases with internal structure γ may be substantially less than $5/3$, but always greater than 1. For an ideal two-dimensional gas, as may be found on a surface for example, γ takes the value 2.

Given the prior formulation of the problem, it is quite easy to see how the result can be generalized to a description of a gas with k degrees of freedom. The only change occurs in the averaging of the temperatures between the intermediate and final steps, that is

$$T_2 = \frac{1}{3}(T_{x,1} + T_{y,1} + T_{z,1}) = T_0 - \frac{2}{kN}pdV$$

leading to

$$pV = cNV^{-2/k}$$

and the generalized adiabatic index for a k -dimensional gas,

$$\gamma = \frac{k + 2}{k}$$

The central idea to this derivation of the equation of state for an adiabatic process is that a gas with k degrees of freedom has k reservoirs of energy. During a change of volume, only one of those degrees of freedom is directly affected by the exterior constraints, and therefore only one component of the energy changes. If the velocities associated with the change in volume are small compared to the sound speed in the gas then we may consider this a quasi-static process such that after each infinitesimal time the system again reaches equilibrium with itself and a single temperature is re-established. This intuitive model also shows quite clearly why the adiabatic index must tend toward unity in the limit of infinite degrees of freedom. Under these conditions, a finite energy due to the work done will be distributed over an infinite number of energy reservoirs, resulting in a vanishing temperature change.