General analysis of two-wave interference

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1. Introduction

In this document we will examine the effects of superposition of waves from two sources. If the waves by some means acquire a different phase (for example, by traversing a different path), then the amplitude of the superposition may be either enhanced (constructive interference), or diminished (destructive interference), an effect generally referred to as "interference".

The phase of each of the component waves is influenced by three factors: the spatial dependence through the kx term, the temporal dependence through the ωt term and the phase factor ϕ. We will consider what happens when each of these terms contributing to the wave phase is varied between sources.

2. Background

To begin, let us consider two sources emitting waves of the form

$$p_1(x,t) = P_1 \cos(k_1 x_1 - \omega_1 t + \phi_1),$$

$$p_2(x,t) = P_2 \cos(k_2 x_2 - \omega_2 t + \phi_2).$$

where the amplitudes $P_1 = P_2$ are positive quantities. At this most general stage of the formulation we allow each wave to have distinct amplitudes, distinct frequencies (ω_1 and ω_2) and distinct phase factors (ϕ_1 and ϕ_2). Because the wavenumber and angular frequency are constrained by the condition $\omega/k = v$, it is necessary only to specify one or the other of ω or k. To proceed we simplify the analysis by introducing a shorthand notation and let the argument of the cosine function be lumped together as a single variable, θ. That is,

$$p_1(x,t) = P_1 \cos(\theta_1),$$

$$p_2(x,t) = P_2 \cos(\theta_2),$$

where,

$$\theta_1 = k_1 x_1 - \omega_1 t + \phi_1$$

$$\theta_2 = k_2 x_2 - \omega_2 t + \phi_2$$

The mathematical representation of the superposition of two waves is that the total fluctuation is just the sum of the two individual waves, that is,

$$p(x,t) = p_1(x,t) + p_2(x,t) = P_1 \cos(\theta_1) + P_2 \cos(\theta_2).$$

The presence of distinct amplitudes as we have accounted for here prevents us from using trig identities to simplify this expression, as was done in the textbook. To make this form more amenable to such analysis we first define the sum and difference amplitudes,

$$P_+ = \frac{P_1 + P_2}{2}$$

$$P_- = \frac{P_1 - P_2}{2}$$

and their inverse relationships,

$$P_1 = P_+ + P_-$$

$$P_2 = P_+ - P_-$$

so that the prior equation for the sum of waves takes the form,

$$p(x,t) = P_+[\cos(\theta_1) + \cos(\theta_2)] + P_-[\cos(\theta_1) - \cos(\theta_2)].$$

Note that in the case of equal amplitudes, the difference term, P_-, vanishes and we recover the simpler formulations already seen. At this point, we are nearly ready to apply our trig identities. Recall,

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$cos(a \pm b) = \cos(a)\cos(b) \mp \sin(\mathrm{a})\sin(b).$$

And by defining new variables α and β as follows,

$$\alpha = a - b$$

$$\beta = a + b$$

with the inverse relationships,

$$a = \frac{\alpha - \beta}{2}$$

$$b = \frac{\alpha + \beta}{2}$$

we can easily derive the following trig relationships from the prior ones by taking sums and differences of the terms on the left-hand-side to derive,

$$cos(\alpha) + cos(\beta) = 2\cos\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right)$$

$$cos(\alpha) - cos(\beta) = 2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$$

Applying these last trig identities to our equation for the sum of waves, we have

$$p(x,t) = 2P_+\cos(\theta_-)\cos(\theta_+) + 2P_-\sin(\theta_-)\sin(\theta_+)$$

where

$$\theta_+ = \frac{\theta_1+\theta_2}{2}$$

$$\theta_- = \frac{\theta_1-\theta_2}{2}$$

represent the sum and difference of the generalized phases of the waves. We anticipate that the terms containing θ_+ will be the wave-like part, and everything else we can lump into the effective amplitude. Recalling that an oscillator of the form

$$f(t) = A\cos(\omega t) + \mathrm{B}\sin(\omega t)$$

has amplitude $(A^2 + B^2)^{1/2}$, the wave superposition expression can be put in the form

$$p(x,t) = P_{eff}\cos(\theta_+ + \phi)$$

where the phase factor takes the form

$$\phi = -\tan^{-1}\left[\frac{P_1-P_2}{P_1+P_2}\tan(\theta_-)\right]$$

and the effective amplitude for our two-wave system is

$$P_{eff}^2 = 4P_+^2\cos^2(\theta_-) + 4P_-^2\sin^2(\theta_-)$$

or, using the identity $sin^2(\theta) + cos^2(\theta) = 1$ and rearranging, we have

$$P_{eff} = (P_1+P_2)\sqrt{1-\frac{4P_1P_2}{(P_1+P_2)^2}\sin^2(\theta_-)}$$

We cannot further simplify this expression and proceed by considering specific cases.

3. The case of equal amplitudes

It is impossible to generate perfect destructive interference (meaning that the effective amplitude goes to zero) except in the case where $P_1 = P_2$. However, even when the amplitudes are not equal interference still occurs and there will be regions of more intense and less intense sound (or light) waves. We can see this in the general amplitude expression, found at the bottom of page 3, by setting the argument inside the square root to zero

$$1 - \frac{4P_1P_2}{(P_1 + P_2)^2}\sin^2(\theta_-) = 0$$

Or, letting x = P_1/P_2 we have, after solving for the sine,

$$\sin^2(\theta_-) = \frac{(1+x)^2}{4x}$$

The right-hand-side (RHS) of this function takes the value 1 when x=1. Because there is only one solution for this value, we know that x=1 either represents a maximum or minimum of the RHS. It must be a minimum because if we choose any other value we find that the RHS takes a value larger than 1. For example, if we choose x=3, the RHS is equal to 4/3.

It is thus proved that perfect destructive interference only occurs when the amplitudes at a given location are equal. When this condition is satisfied (using P_0 to represent the amplitude of both waves), we find that the effective amplitude expression takes a simpler form:

$$P_{eff} = 2P_0\sqrt{1 - \sin^2(\theta_-)} = 2P_0\cos(\theta_-)$$

This is exactly the result derived in lecture and in the textbook for superposition of equal amplitude waves when $k\Delta x/2$ is substituted for θ_-. In the following we will consider this simplified expression for the case of equal amplitude waves.

4. Sources with different frequencies

Summing of two wave sources with different frequencies produces the effect of "beats". When the frequency difference between two sources is small compared to the mean frequency, superposition of such waves produces the sensation of a low frequency, periodic fluctuation of the amplitude of the sound waves. With waves of different frequencies (and hence, different wavenumbers) and possibly phase factors, the θ_- term takes the form, but assuming that the sources are at the same location (x_0), we have:

$$\theta_- = (k_1 - k_2)x_0 - (\omega_1 - \omega_2)t + (\phi_1 - \phi_2)$$

Because we are more concerned with the temporal variation in this problem, we will simplify this expression and lump the spatial dependence in with the phase constants with the following definition

$$\Delta\theta = (k_1 - k_2)x_0 + (\phi_1 - \phi_2)$$

So that we have

$$P_{eff} = (P_1 + P_2)\sqrt{1 - \frac{4P_1P_2}{(P_1 + P_2)^2}\sin^2\left(\frac{\Delta\omega\, t + \Delta\theta}{2}\right)}$$

This represents modulation of the effective amplitude at a frequency of $\Delta\omega/2$, which is called the modulation frequency, ω_{mod}. To our ear, the more meaningful quantity is the frequency with which a minimum of the effective amplitude is reached, which occurs every time sine takes the values ±1, or twice per cycle. Therefore the beat frequency is $\omega_{beat} = 2\omega_{mod} = \Delta\omega$.

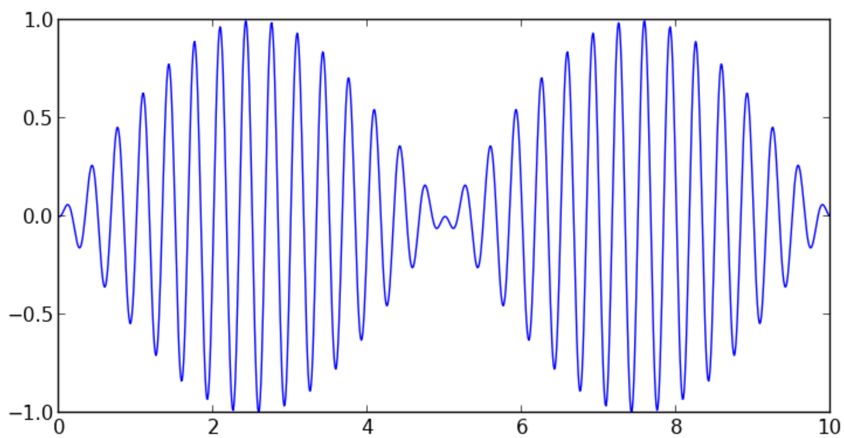

Figure 1: Resulting waveform from the addition of equal amplitude waves at frequencies of 3.0 Hz and 3.2 Hz. The resulting average frequency is 3.1 Hz, with a modulation frequency of 0.1 Hz. The beat frequency is 0.2 Hz, since once every 5 seconds we develop a lull in the amplitude.

5. Spatially separated sources of the same frequency and same phase factor

This case represents the interference resulting from waves emitted by two speakers, or in the case of light waves, the two-slit diffraction pattern. We consider the additional constraint that a single oscillator drives the two sources so that they have the same phase factors. With these conditions, the sum and difference phases take the form

$$\theta_+ = kx_{avg} - \omega t + \phi$$

$$\theta_- = \frac{k(x_1 - x_2)}{2} = \frac{k\Delta x}{2}$$

where $x_{avg} = (x_1 + x_2)/2$ is the average distance from the sources at the measurement location. In the case of equal amplitudes ($P_1 = P_2 = P_0$), we have a simpler expression, which does exhibit perfect destructive interference,

$$P_{eff} = 2P_0 \cos\left(\frac{k\Delta x}{2}\right)$$

The only computational exercise left at this point is to calculate Δx for a given configuration, a purely geometric problem. This can be easily done numerically to any desired accuracy, or we can use various approximations such as $\Delta x \approx d \sin(\theta)$, where d is the spacing between the sources.

We now ask what conditions Δx must satisfy so that we achieve destructive interference in the case of equal amplitudes. This is equivalent to finding the values of the argument for which cosine vanishes, leading to the condition

$$\frac{k\Delta x}{2} = (2n+1)\frac{\pi}{2}$$

or equivalently,

$$\Delta x = (2n+1)\frac{\lambda}{2}$$

where n can be any integer (positive, negative or zero). Note that $2n+1$ produces only the odd integers, so we could, without loss of generality, replace $2n+1$ with an integer $j = \pm 1, \pm 3, \pm 5, \ldots$ and so on. The values of allowable n for a particular case can be determined by considering the maximum possible Δx. For the case of the double-slit experiment, we found $\Delta x \approx d \sin(\theta)$, where θ is properly defined as the angle between the midpoint of the slits and a circle of radius r in the plane of the sources. This expression clearly shows that the maximum possible Δx is d, and occurs when $\theta = \pi/2$, which makes sense.

Setting $\Delta x = d$ we can solve for the maximum n for destructive interference. Doing so gives

$$\max(n) \leq \frac{d}{\lambda} - \frac{1}{2}$$

where we have defined the function max(n) to mean "n can be the largest integer less than or equal to this value". For example, if we consider the case of a 40 Hz sound wave (in air at v=340 m/s), then $\lambda = 8.5$ m, with two sources separated by 20 m, then we have max(n) = 1.85, so that $n_{max} = 1$. The total number of destructive interference regions to the right of center is 2 (n=0 and n=1, or equivalently, $j=+1$ and $j=+3$), so that the total number of interference fringes is 4 (as seen by the dark bands in Figure x).

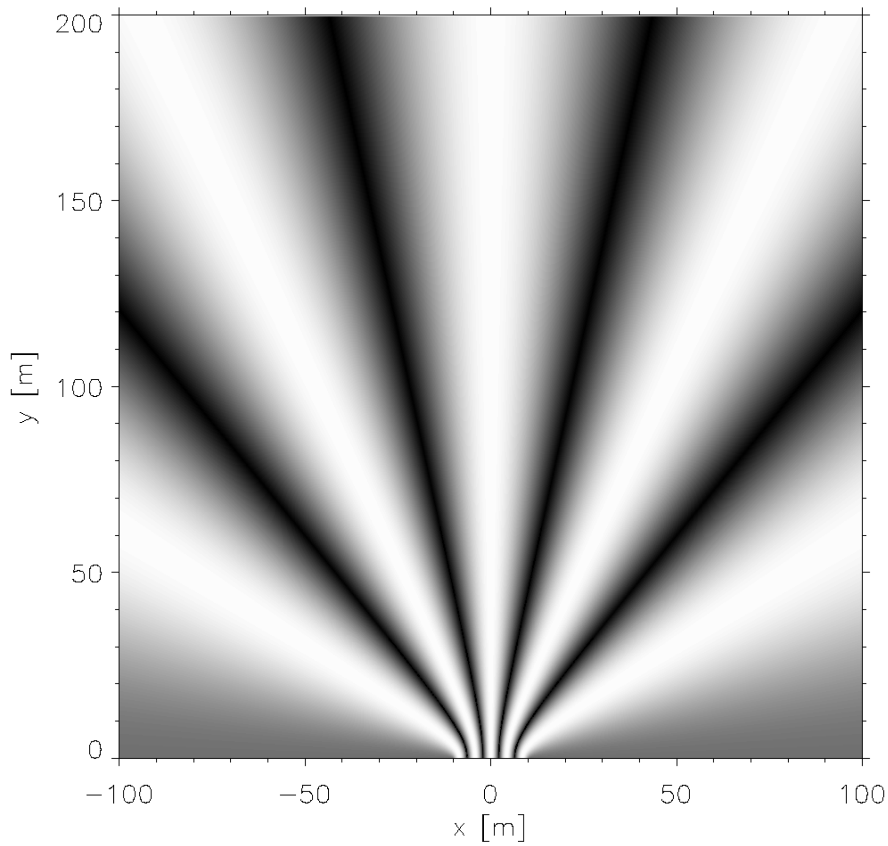

Figure 2: Interference pattern for sound waves with f = 40 Hz in air with a sound speed of 340 m/s. The sources are placed at +/- 10 m along the x-axis. For this case we have five complete constructive interference bands and four complete destructive interference bands. The regions along the x-axis are neither complete destructive or constructive interference and do not count toward these totals.

We can repeat this analysis and now solve for the points of constructive interference. This is done by setting

$$\cos\left(\frac{k\Delta x}{2}\right) = \pm 1$$

which leads to

$$\frac{k\Delta x}{2} = n\pi$$

Again setting $\Delta x = d$, we can solve for the number of constructive interference fringes:

$$\max(n) \leq \frac{d}{\lambda}$$

Using the same values as in the prior example, the RHS of this expression is equal to 2.35, which means that *n* can take the values 0, 1 or 2. Given that the *n=0* fringe occurs right in the center of the pattern, the total number of constructive interference fringes is 5, as can be seen in Figure 2.

It is interesting to also examine the general solution ($P_1 \neq P_2$) because the sound amplitude from a source tends to decrease inversely proportional to the distance from the source. Thus, even if two sources are established to have the same intensity at one point, it will not be true in general that the amplitudes are equal elsewhere.

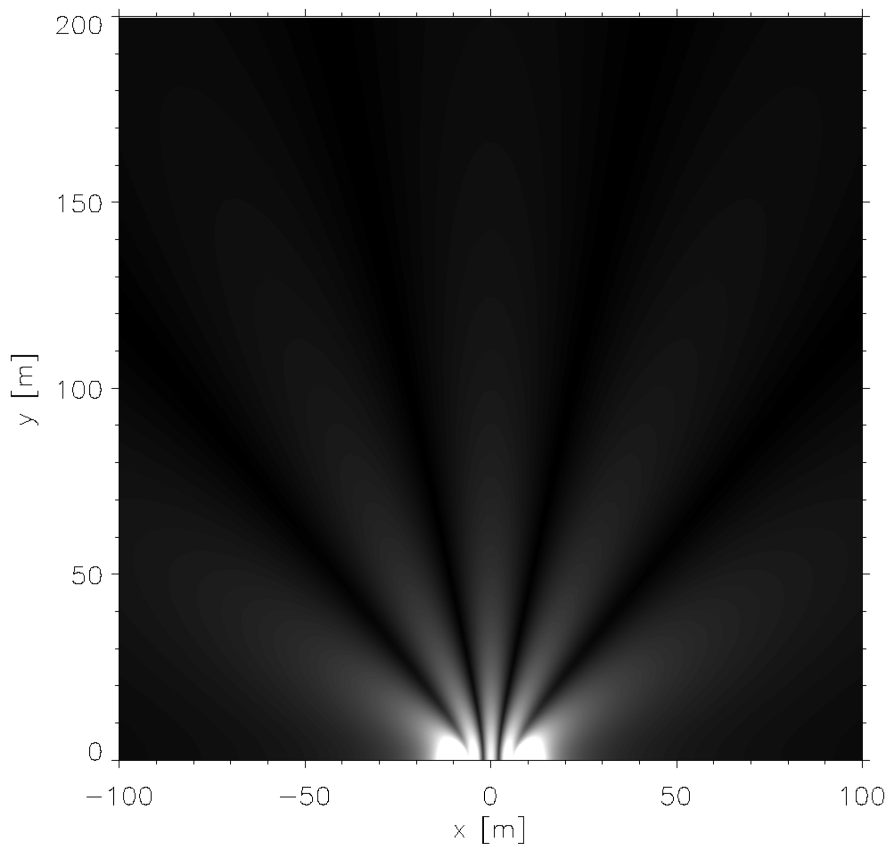

Figure 3: Interference pattern for the case of f = 40 Hz sound waves in air with a sound speed of 340 m/s. This case illustrates the pattern we expect if the amplitude of the waves from each source falls off inversely proportional to the distance from the source. Comparing to Figure 2, we see that the arrangement of the regions of more-or-less constructive or destructive interference patterns does not change significantly, though the relative amplitude does.

6. Spatially separated sources of the same frequency but different phase factor

The analysis of the case of two spatially separated sources driven at the same frequency but with a phase difference between the two drivers follows the prior argument, only now $k\Delta x$ is replaced by $k\Delta x + \Delta\phi$. This has the effect of shifting the lines of destructive and constructive interference to the left or right, depending on the sign of the phase shift. The condition on Δx for constructive interference is now

$$\frac{k\Delta x + \Delta\phi}{2} = n\pi$$

where

$$\Delta\phi = \phi_1 - \phi_2.$$

Or, using the approximation $\Delta x \approx d \sin(\theta)$ we can solve for the angles of the lines of constructive interference.

$$\sin(\theta) = \left[n - \frac{\Delta\phi}{2\pi}\right]\frac{\lambda}{d}$$

We are often concerned with the position of the central (*n=0*) beam. This relation tells us that

$$\sin(\theta_0) = -\frac{\Delta\phi}{2\pi}\frac{\lambda}{d}$$

If we want to sweep a beam through a specified angle, this formula tells us the relationship between that angle and the phase difference required to produce this effect. Many industrial and experimental techniques work on this principle, including phased-array radar, directional ultrasound and radio frequency heating of thermonuclear plasmas.

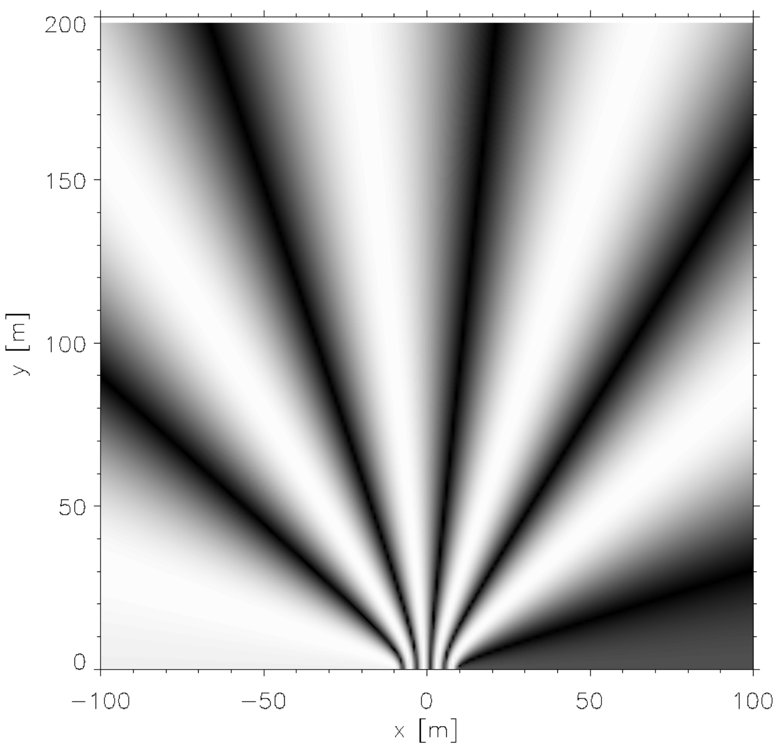

Figure 4: Same parameters as in Figure 2 (f = 40 Hz, v = 340 m/s) except that the phase of the right speaker is advanced by $\pi/2$ relative to the left speaker, that is, $\Delta\phi = -\pi/2$. Under these conditions the central maximum is steered to a positive angle of approximately 6°, in agreement with the above plot. Since the central maximum is displaced to the left by about 20 meters at a distance of 200 meters, this angle is approximately $\tan^{-1}(20/200) \approx 6°$.