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Measurement of the length-period relationship for a simple pendulum and inference of the local gravitational acceleration

Introduction

A set of measurements of a simple pendulum system have been conducted to test the relationship that the square of the period is proportional to the length of the pendulum and independent of the realse angle,

$$T^2 = \frac{4\pi^2}{g}L \qquad \text{(Eq. 1)}$$

By measuring the period for a range of release angles and pendulum lengths the model described by Eq. 1 has been tested. We found weak dependence on angle for release angles less than 40 degrees and linear dependence on length, in agreement with the model. From this data the local gravitational constant, g, was determined to be close to 9.75 m/s^2.

Experimental setup

A simple pendulum composed of a long, thin synthetic fiber string and a metal bob weighing approximately 100 grams was suspended from a horizontally oriented clamp. Care was taken to clamp the string in a configuration that maintained a constant length pendulum throughout the oscillation. A long length of string was used and the knot on the bob tied close to the bob, and tied only once, with observations of increasing length of pendulums performed by playing out more string through the clamping mechanism.

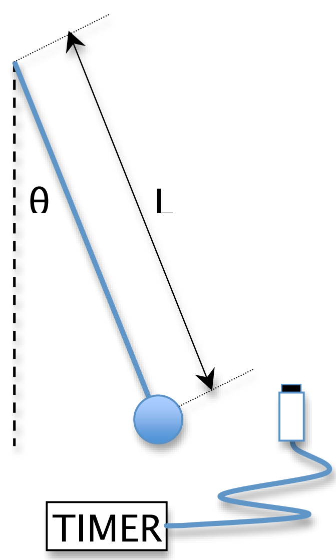

Fig. 1: Sketch of the experimental apparatus.

Measurement of the pendulum length was made between the bottom of the clamp and the center of the bob by visual reference to a meter stick placed alongside the pendulum when stationary. Estimated uncertainty in this measurement is 1 mm.

The pendulum period was measured by a timer that is stopped and started by a game-show-like buzzer operated under human control. Systematic errors in timing were minimized by using the same person to conduct all experiments, having the timer operator release the pendulum and operate the buzzer simultaneously, and by making multiple measurements under the same conditions. For the experiments testing the relationship between period and length a release angle of 20 (+/- 3) degrees was used for all trials to eliminate a possible angular dependence on period. Based on post-facto analysis of period measurements, the estimated uncertainty in the timing control was no larger than 100 ms. Given that the measured times were for ten cycles of the pendulum, this timing uncertainty translates to a maximum uncertainty in the period of 10 ms. Because we are unable to account for possible systematic errors (bias) in timing measurements, we assume for this analysis that there is no bias.

Data and analysis

Part 1: Period dependence on release angle

We present first the data from the measurements of period as a function of release angle. These experiments were conducted with a pendulum length of 86.5 cm.

angle [deg]	period [sec]
10	1.84
20	1.87
30	1.88
40	1.88
50	1.91
60	1.94
70	1.98
80	2.02

Table 1: Measurements of period as a function of release angle. Data shown is the actual measurement of period (10 cycles) divided by 10.

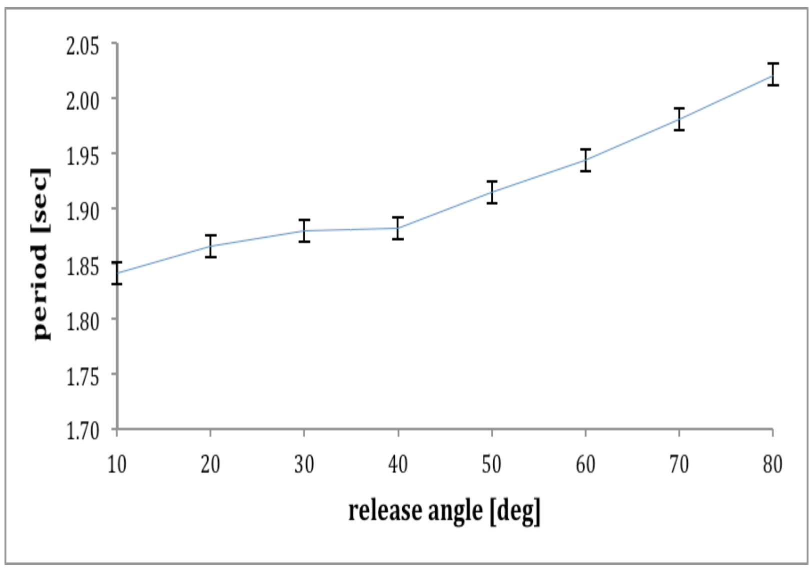

Fig. 2: Period vs. angle for the data from Table 1. There appears to be a transition in the response near an angle of 40 degrees, above which the period increase more rapidly with angle.

The Taylor expansion used in deriving the SHO equation for the simple pendulum, $\sin(\theta)\approx\theta$, is most accurate for small angles. The apparent linear dependence of period on angle for angles less than 40 degrees is somewhat surprising. Our hypothesis is that some systematic error is present in the case of the 10 degree measurement. Were this one value excluded from the presentation, we would observe a nearly constant period-angle relationship in the small angle regime, at least within the error bars of the measurements.

Part 2: Period dependence on length

With the exception of a likely erroneous measurement at a release angle of 10 degrees, the results of part 1 of this experiment suggest that relatively angle-independent measurements of period can be made for angles less than approximately 40 degrees. For the remaining studies, we use a release angle of 20 degrees as a compromise between an ideally small angle and a large-enough angle so that timing errors are not too large.

length [cm]	trial #1	trial #2	trial #3	trial #4	trial #5	period [sec]	std. dev. [sec]
15.1	7.87	7.80	7.83	7.82	7.84	0.783	0.002
25.4	10.16	10.15	10.08	10.14	10.16	1.014	0.003
34.7	11.90	11.90	11.86	11.82	11.92	1.188	0.004
47.1	13.81	13.99	13.88	13.80	13.87	1.387	0.007
58.5	15.33	15.41	15.44	15.33	15.35	1.537	0.004
72.4	17.08	17.03	17.09	17.08	17.17	1.709	0.005
86.5	18.62	18.67	18.58	18.77	18.62	1.865	0.007

Table 2: Data for the studies of the dependence of period on pendulum length. The multiple trials listed are measurements of the time for 10 full cycles, measured in units of seconds. The last two columns present the sample mean and standard deviation for these sets of measurements.

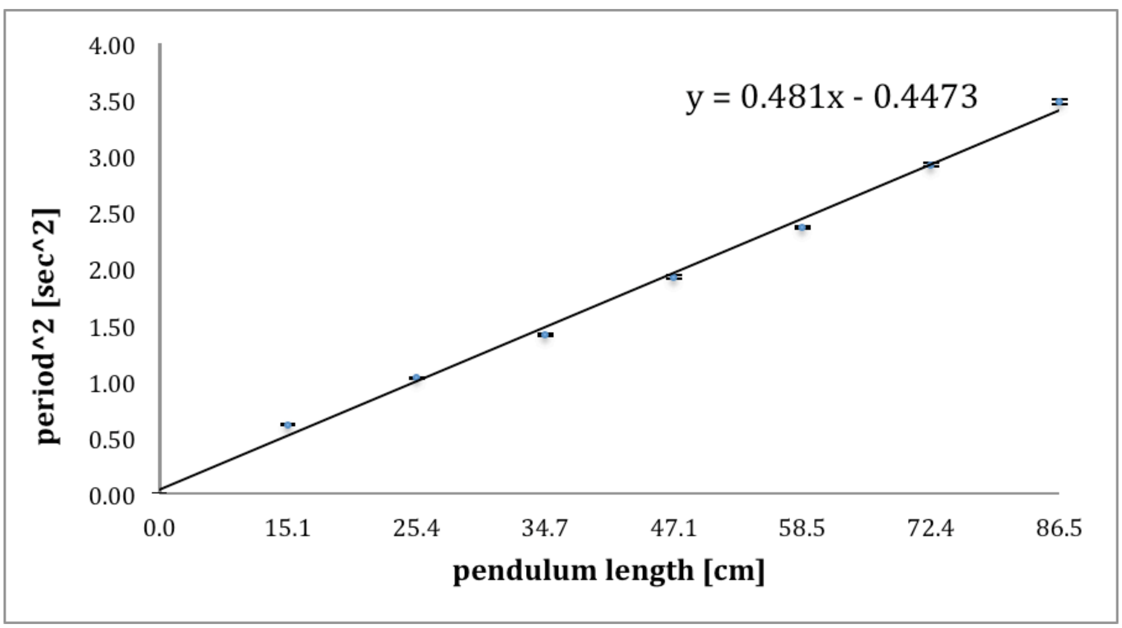

Fig. 3: Plot of the square of the period exhibits with pendulum length. The error bars for the measurements are in some cases smaller than the deviation of a measurement from the linear fit, yet, overall the system seems to have a nearly linear relationship.

Part 3: inference of the local gravitational acceleration from period measurements

A small angle approximation to the force analysis for a simple pendulum provides a relationship between period and pendulum length, as in Eq. 1. This equation may be solved for g as a function of the measured period and length.

$$g = 4\pi^2 \frac{L}{T^2} \qquad \text{(Eq. 2)}$$

The uncertainty in this method of measurement arises from two sources: uncertainty in the length of the pendulum and uncertainty in the measurement of the period.

$$\frac{\Delta g}{g} = \sqrt{\left(\frac{\Delta L}{L}\right)^2 + 4\left(\frac{\Delta T}{T}\right)^2} \qquad \text{(Eq. 3)}$$

The uncertainty in the length was estimated to be 1 mm, for all cases. The largest error arising from uncertainty in the length then occurs for the smallest length of 15.1 cm, giving a fractional uncertainty of roughly 7×10^{-3}. The timing uncertainty was largest in the case of 47.1 cm, with a fractional uncertainty of roughly 5×10^{-3}. We use these values to form an approximate fractional uncertainty for g of order 1×10^{-2} for each measurement.

length [cm]	period [sec]	g (m/s^2)
15.1	0.783	9.72
25.4	1.014	9.76
34.7	1.188	9.71
47.1	1.387	9.67
58.5	1.537	9.77
72.4	1.709	9.79
86.5	1.865	9.82

Table 3: Inferred values of g as a function of measured pendulum length and period. The average value of g from these measurements is 9.75 m/s^2.

Conclusions

This study used measurements of the period of a simple pendulum to look at three issues: the angular dependence of the period, the relationship between period and pendulum length and lastly, an inference of the local gravitational acceleration from these measurements.

On the first issue, the experiments showed a stronger dependence of the period on angle above approximately 45 degrees. This roughly correlates with an intuitive division between dynamics which are either more horizontal or more vertical. Based on analysis of the forces actin on a simple pendulum, we expect that the periods measured at angles less than 45 degrees should be roughly independent of angle, suggesting that the measurement made at 10 degrees is rather anomalous in the set and is likely the result of human error in trying to identify the position at which the bob stops. Further studies are warranted to track down isolate this effect.

One the second count, that of exploring the relationship between the period and length, we find generally a linear agreement over pendulum lengths from 15 cm to 85 cm. The deviations of the individual measurements from the linear fit are in some cases larger than the error bars, a worrisome effect. It may be that a consistent bias on the part of the timer was responsible for these errors. Repeating these measurements with an optical or inductive trigger to more accurately measure the period of the pendulum would be the best method for resolving these discrepancies.

And lastly, our inference of the local gravitational acceleration produced an average value of 9.75 m/s^2, a value in good agreement with the commonly used value of 9.8 m/s^s, with a fractional difference of about $5x10^{-3}$. We believe this to be a quite decent measurement of g given the simple tools used in this study. Local geologic variations can produce deviations in g of order a few tenths of a percent, substantially smaller than our error, suggesting that substantial systematic errors remain.