The tale of the fish and the fisher-people

Physics 203, Spring 2020

E. M. Edlund

Creating a model for population interactions

- Basic concepts:
  1. Fish population (P_F):
     1. There is some maximum fish population that the oceans can support.
     2. The more fish people eat, the fewer fish there are.
     3. The fish population regenerates on some characteristic time scale (a fish generation).

  2. Human fisher-people population (P_H):
     1. People need to eat fish to survive and reproduce.
     2. The growth of the human population depends on how many people there are, and also how many fish they can harvest.
     3. The population grows on some characteristic time scale (a human generation).

Building a simple model first

- Let's begin with a simpler idea first:
  - How do we model something simple, like a single bacterial population, in which there is an unlimited food supply and each bacteria lives forever?

- Each bacteria grows and then divides into two bacteria over a generational time scale.
  - We start with an initial bacteria population of P_0 .
  - After one generation the population doubles to $P_1 = 2P_0$.
  - After another generation the population again doubles, giving $P_2 = 2P_1 = 4P_0$.
  - In general, the n^{th} generation will have a population of $P_n = 2^n P_0$.

Building a simple model first

- This is what we call "geometric" growth. It is very similar to exponential growth.

- In fact, we can put this growth equation in exponential form.

- To do so we use the natural logarithm: $\ln(2^x) = x \ln(2)$

- Exponentiating we have: $2^x = \exp(x \ln(2))$

- This allows us to reformulate our population model using the exponential function:

$$\begin{aligned} P_n &= \exp(n \ln(2))\, P_0 \\ &= P_0 \exp(n \ln(2)) \end{aligned}$$

Building a simple model first

- We might want to consider what happens as we "zoom out" to large times and consider a continuous variation of the population in time.

- To do so, we note that the time at generation *n* is $t_n = n\tau$ where τ is the time associated with one generation.

- Flipping that relation around we have: $n = \dfrac{t_n}{\tau}$

- The continuous population model is: $P(t) = P_0 \exp\left(\dfrac{\ln(2)\, t}{\tau}\right) = P_0 \exp\left(\dfrac{t}{\tau_{eff}}\right)$

Building a simple model first

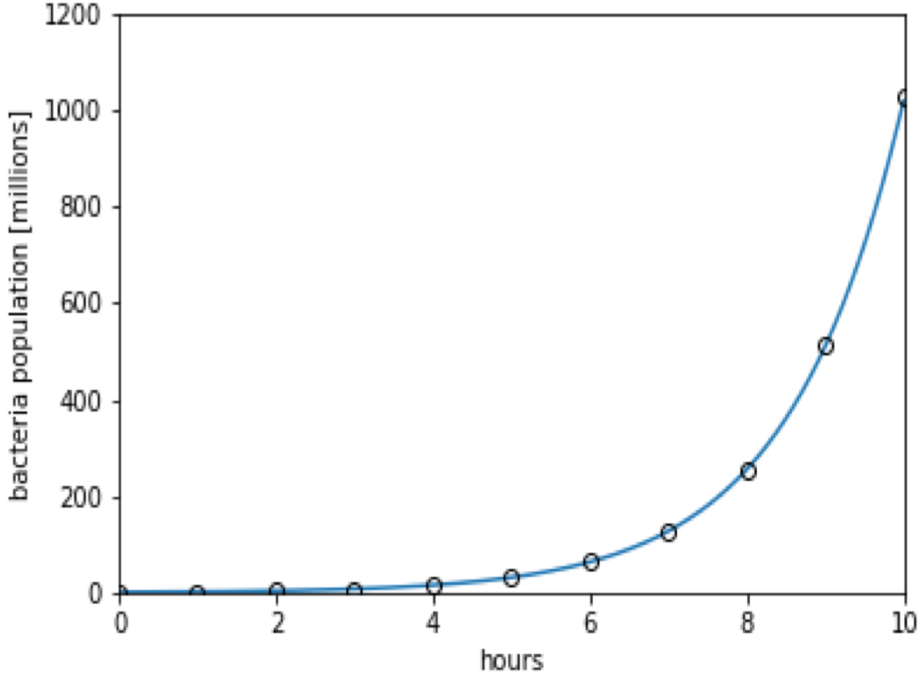
- We can identify the effective time scale for **exponential** growth as:

$$\tau_{eff} = \frac{\tau}{\ln(2)}$$

- For example, if a bacteria population doubles once per hour, then we have

$$\tau_{eff} = \frac{60 \text{ minutes}}{0.693} \approx 87 \text{ minutes}$$

$P_0 = 1$ million



What happens if the dividing is increased?

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$$P_n = 3^n \, P_0$$

$$P_n = P_0 \exp(n \ln(3))$$

$$P(t) = P_0 \exp\left(\frac{\ln(3)\, t}{\tau}\right)$$

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$$\tau_{eff} = \frac{\tau}{\ln(3)}$$

- Using the same conditions as in the previous example, we have

$$\tau_{eff} = \frac{60 \text{ minutes}}{1.10} \approx 55 \text{ minutes}$$

The solution is still an exponential, the only difference being that the time scale for growth is reduced, as we anticipated.

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- We can find the differential equation describing the population growth by taking a derivative:

$$P(t) = P_0 \exp\left(\frac{t}{\tau_{eff}}\right) \quad \Rightarrow \quad \frac{dP}{dt} = \frac{P_0}{\tau_{eff}} \exp\left(\frac{t}{\tau_{eff}}\right) \quad \Rightarrow \quad \frac{dP}{dt} = \frac{1}{\tau_{eff}} P$$

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- This last equation tells us that unlimited exponential growth occurs when the rate of change of population is proportional to the population.

A limited bacterial growth model

- If we want to impose a limit on the population growth, we can go in and tinker with the growth equation.

$$\frac{dP}{dt} = \frac{1}{\tau_{eff}}\{\text{a function that limits growth}\}P$$

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- There are an infinite number of functions we could use, but we might choose something simple:

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when $P \ll P^{max}$ $\quad \frac{dP}{dt} \approx \frac{1}{\tau_{eff}}P$

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when $P \ll P^{max}$ $\quad \frac{dP}{dt} \approx \frac{1}{\tau_{eff}}P$

when $P \approx P^{max}$ $\quad \frac{dP}{dt} \approx 0$

when $P > P^{max}$ $\quad \frac{dP}{dt} < 0$

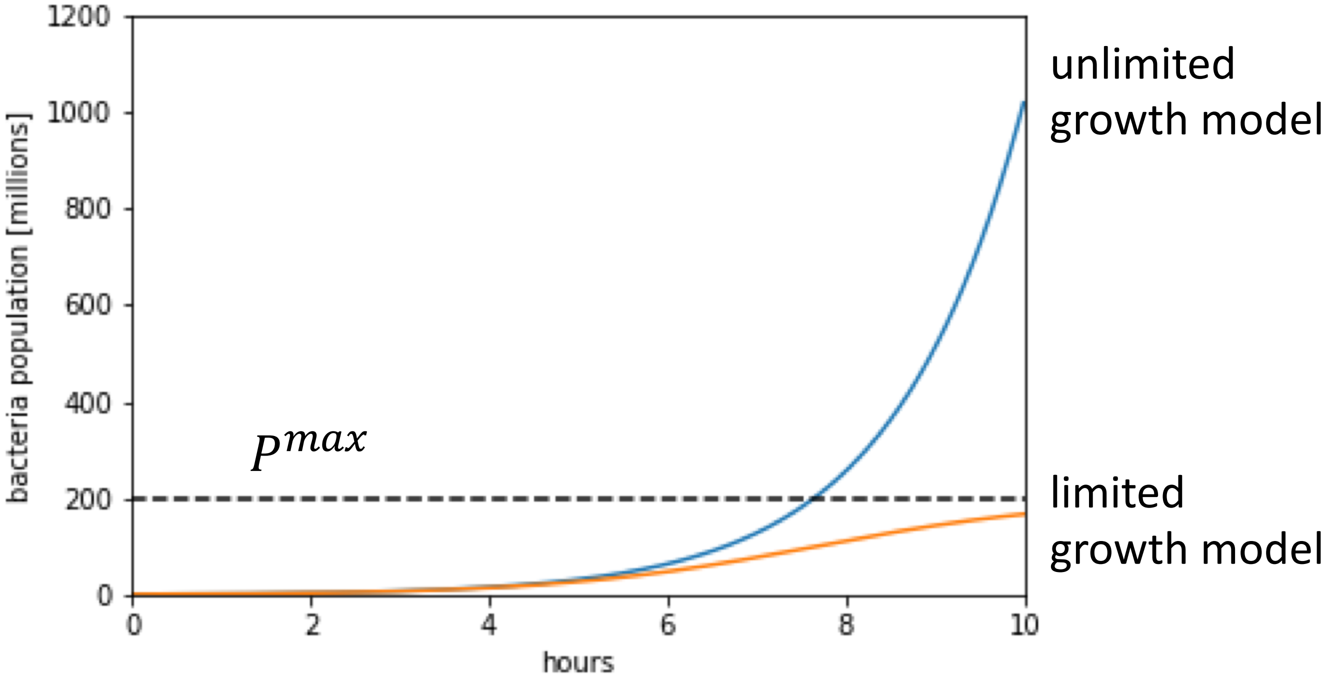
A limited bacterial growth model

- We can integrate the differential equation on the previous page to solve for the population as a function of time.
- Let’s consider a limited growth model the following parameters:

$$\tau_{eff} = 87 \text{ minutes}$$

$$P^{max} = 200 \text{ million}$$

$$P_0 = 1 \text{ million}$$



Modeling a predator-prey system

- We almost have all of the elements in place to consider the fate of the fish and the fisher-people. We just need to model the interaction between the populations.

- We really need to add only one thing, which is that the fish population is reduced when people catch fish.

Modeling a predator-prey system

- We almost have all of the elements in place to consider the fate of the fish and the fisher-people. We just need to model the interaction between the populations.
- We really need to add only one thing, which is that the fish population is reduced when people catch fish.
- Imagine that each person needs 3 fish per day to survive, then the loss of fish (per day) would be 3 times P_{humans}.

$$\frac{d}{dt}P_{fish} = \frac{1}{\tau_{eff}}\left(1 - \frac{P_{fish}}{P_{fish}^{max}}\right)P_{fish} - RP_{humans}$$

The loss of fish is proportional to P_{humans}. The constant ***R*** represents how many fish each person needs to survive.

The fish and fisher-people model

- We make a separate equation for each population:

$$\frac{d}{dt}P_{fish} = \frac{1}{\tau_{fish}}\left(1 - \frac{P_{fish}}{P_{fish}^{max}}\right)P_{fish} - RP_{humans}$$

$$\frac{d}{dt}P_{humans} = \frac{1}{\tau_{humans}}\left(1 - \frac{P_{humans}}{P_{humans}^{max}}\right)P_{humans}$$

- We note two things:
  - Each population has its own generational time-scale.
  - The human population equation does not have a second term on the RHS.

One final detail…

- Whereas the maximum fish population is set by the capacity of the oceans, we need to ask what sets the maximum human population?

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- Whereas the maximum fish population is set by the capacity of the oceans, we need to ask what sets the maximum human population?
- Is there an absolute maximum?
- Maybe it makes more sense to say that the maximum human population is not fixed but that it varies, and at any time depends on the quantity of fish in the oceans at that time.
- We implement this with the following equation: $P_{humans}^{max} = \mathrm{c}\, P_{fish}$

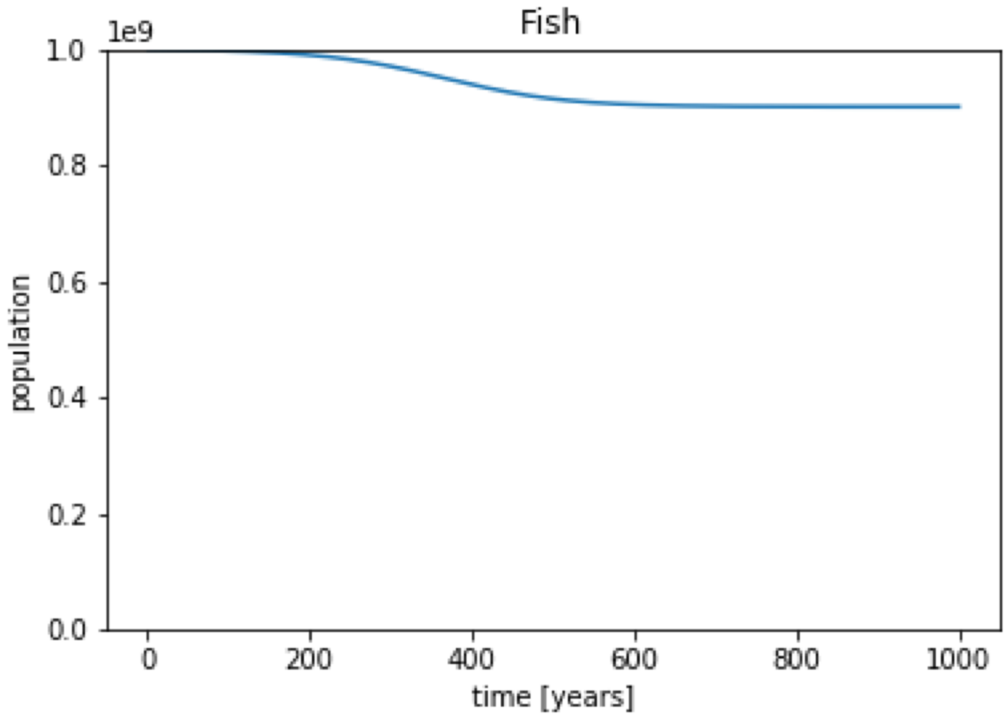
Putting it together

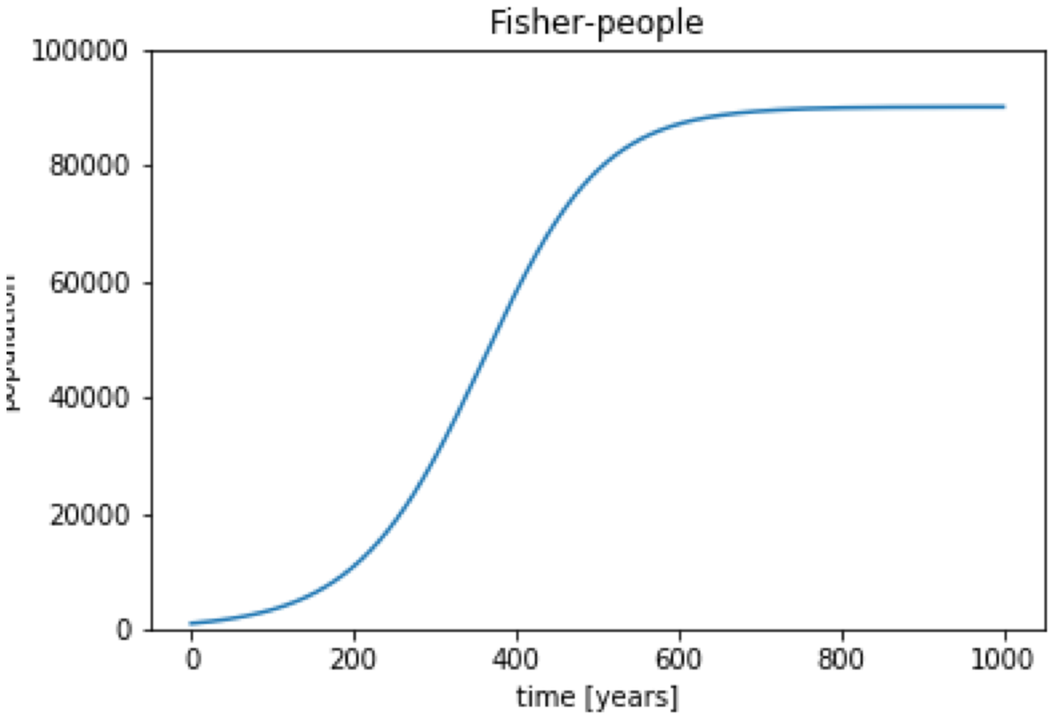
- The next series of slides shows the resulting trends in population growth for the fish and the fisher-people for different model parameters.

- The solutions are integrated on a computer as there is not a simple solution as in the case of bacterial growth.

- The time scale for the integration is considered to be one year. Thus, if each person needs 3 fish per day, then we have $R = 3 \times 365 = 1095$.

The successful population





$P_{0,fish} = 1\times10^9$

$\tau_{fish} = 1$ year

$P_{fish}^{max} = 1\times10^9$

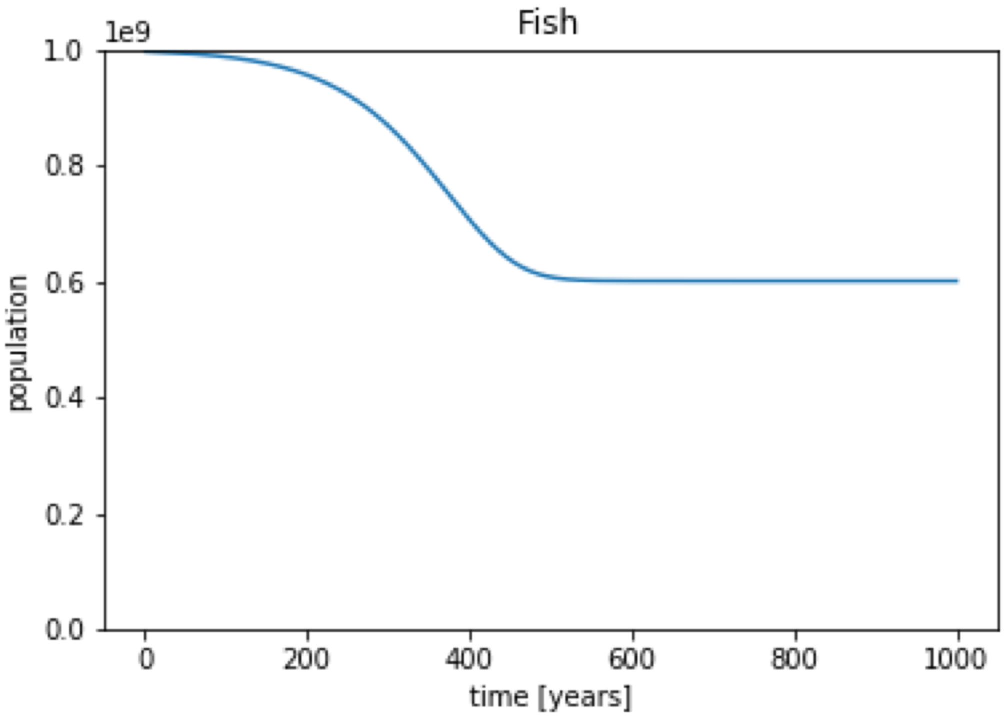
$R = 1000$

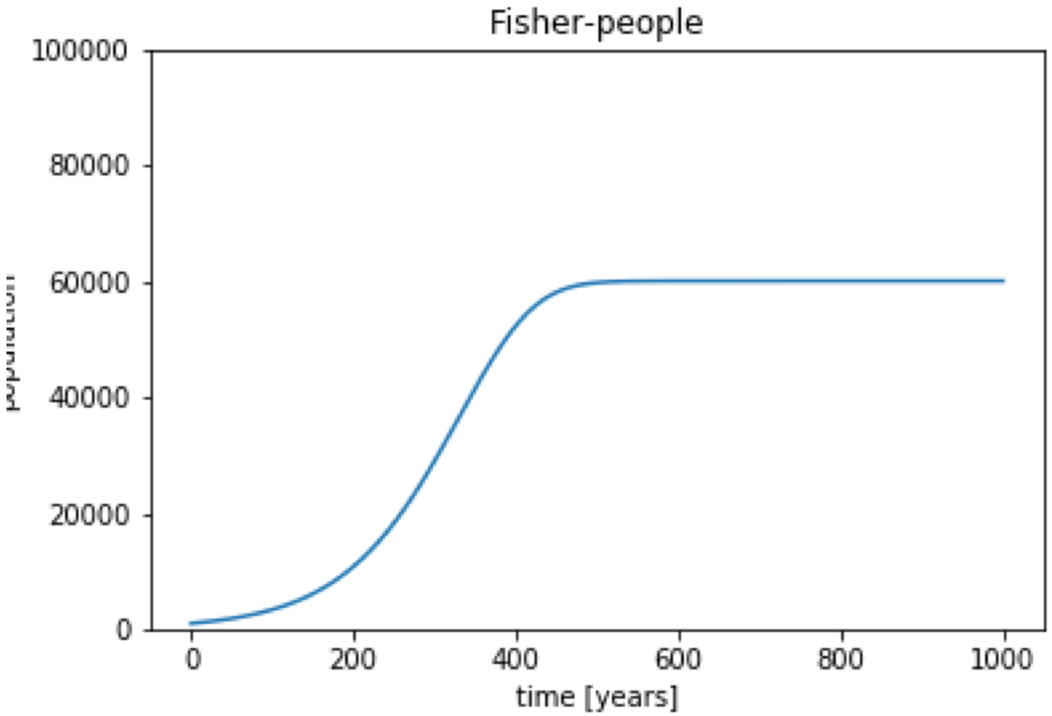
$P_{0,humans} = 1000$

$\tau_{humans} = 80$ years

$c = 0.0001$

The hungry population





$P_{0,fish} = 1 \times 10^9$

$\tau_{fish} = 1 \text{ year}$

$P_{fish}^{max} = 1 \times 10^9$

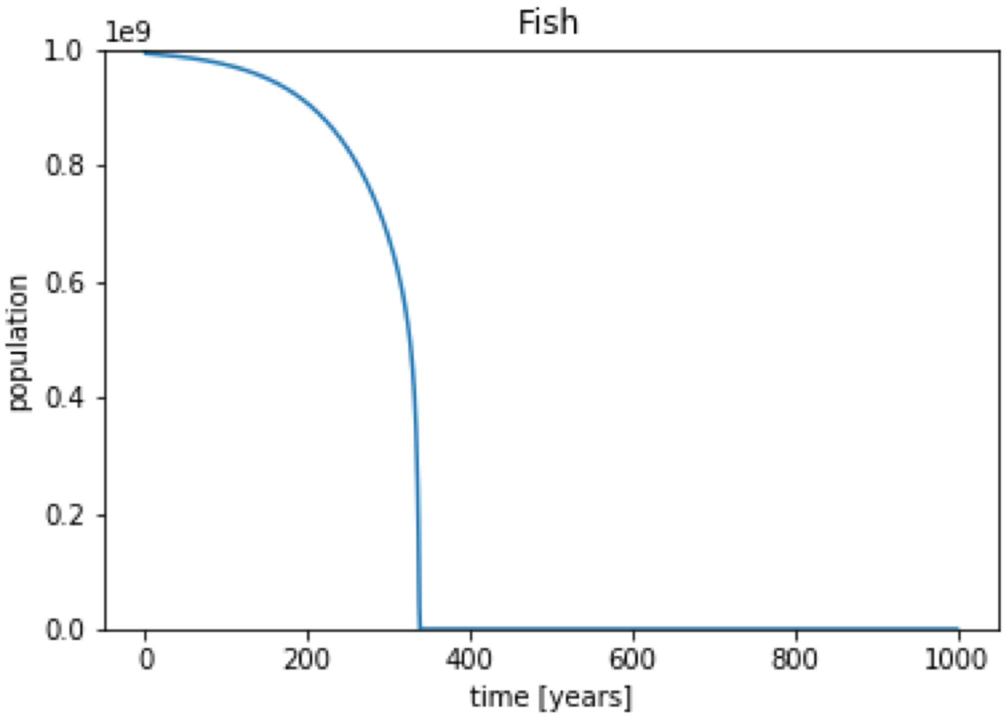
$R = 4000$

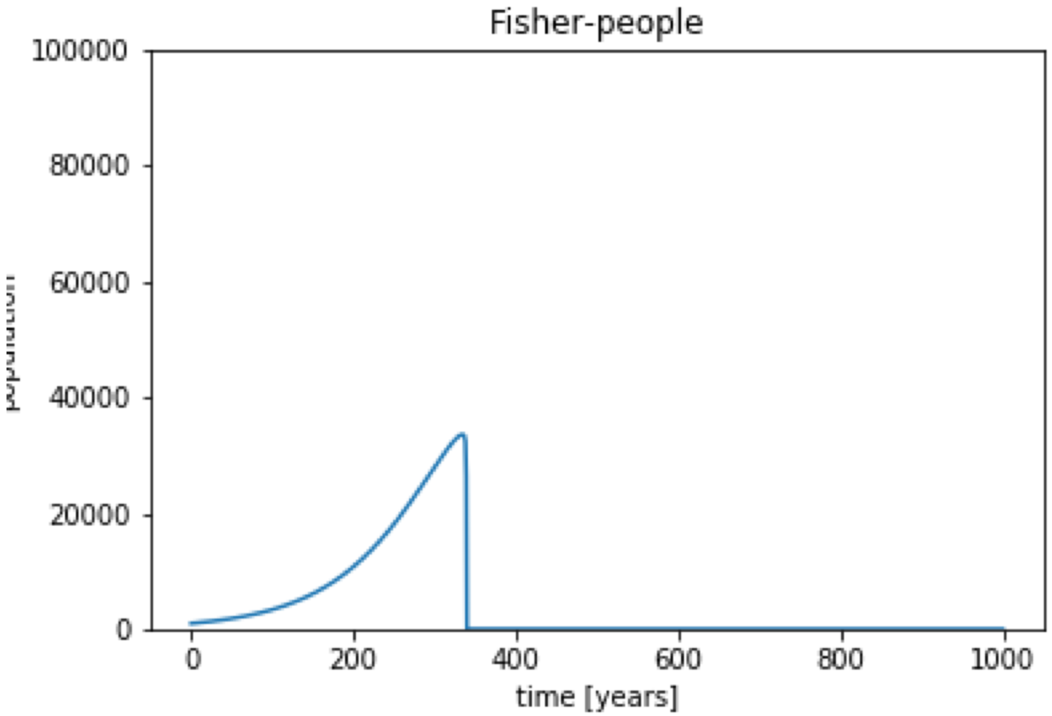
$P_{0,humans} = 1000$

$\tau_{humans} = 80 \text{ years}$

$c = 0.0001$

The gluttons





$P_{0,fish} = 1\times10^9$

$\tau_{fish} = 1 \text{ year}$

$P_{fish}^{max} = 1\times10^9$

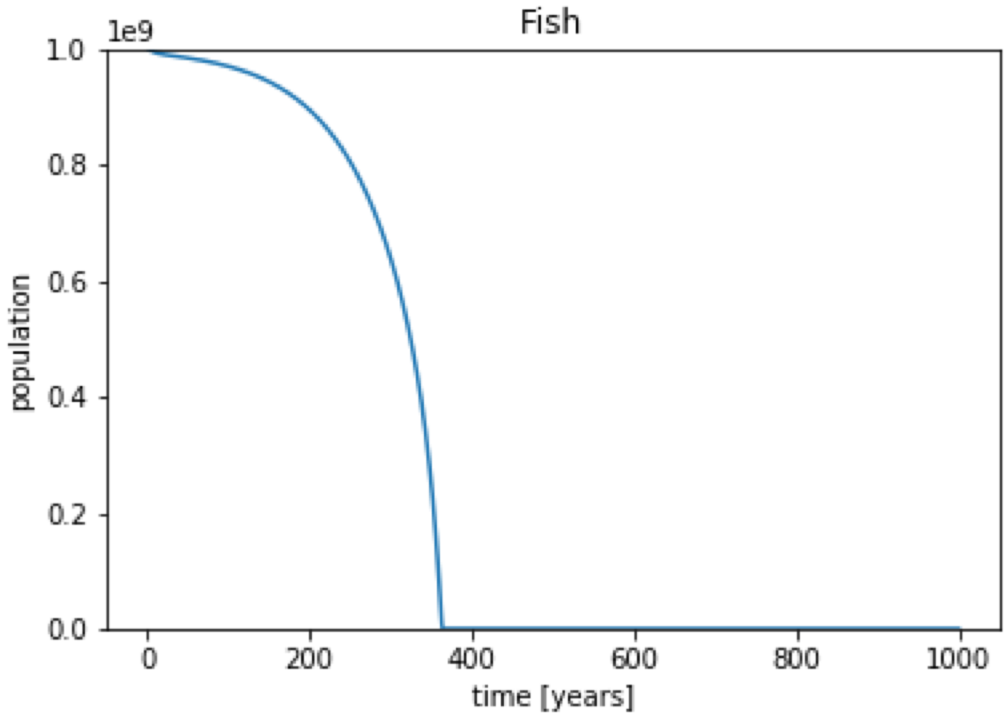
$R = 8000$

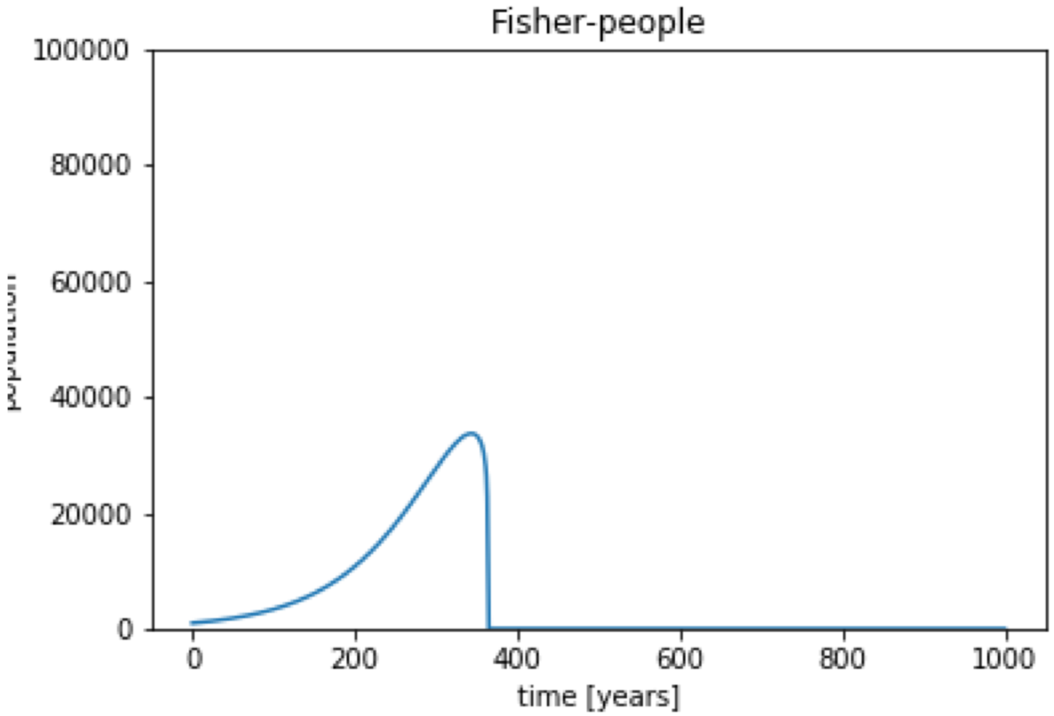
$P_{0,humans} = 1000$

$\tau_{humans} = 80 \text{ years}$

$c = 0.0001$

Long fish generation





$P_{0,fish} = 1\times10^9$

$\tau_{fish} = 10$ years

$R = 1000$

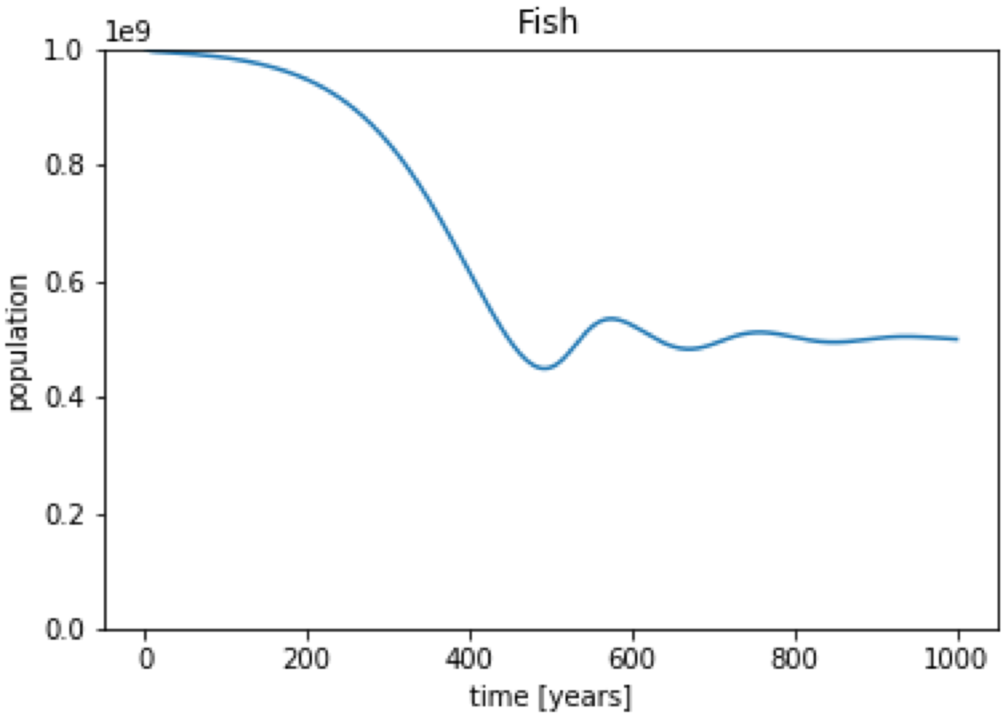
$P_{fish}^{max} = 1\times10^9$

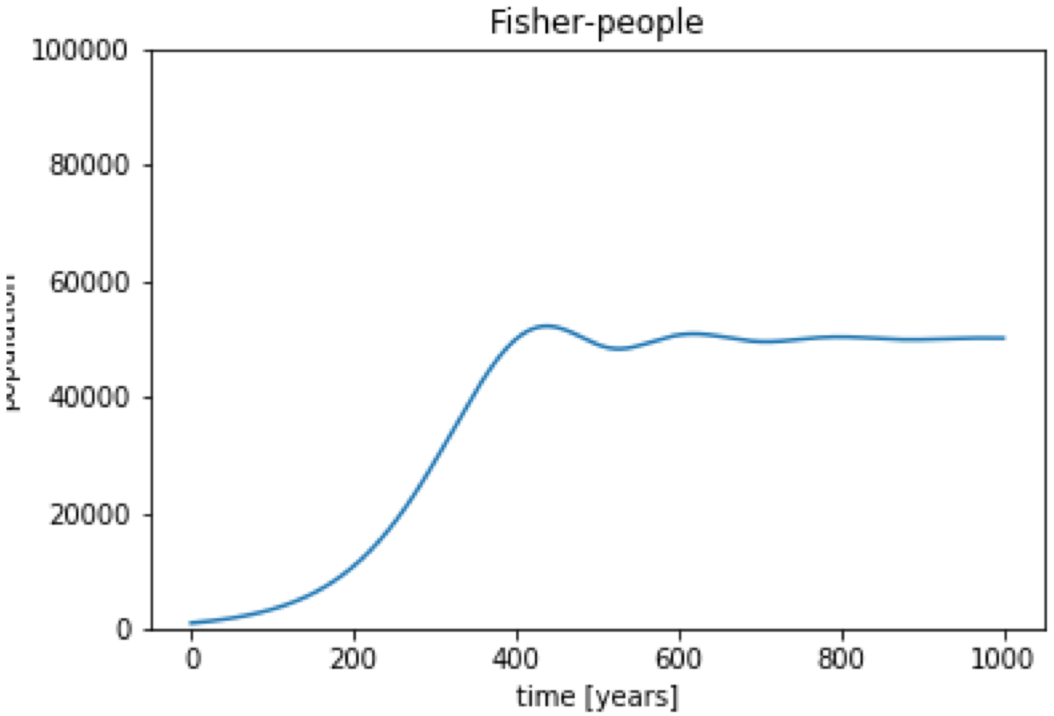
$P_{0,humans} = 1000$

$\tau_{humans} = 80$ years

$c = 0.0001$

Intermediate fish generation





$P_{0,fish} = 1\times10^9$

$\tau_{fish} = 5 \text{ years}$

$P_{fish}^{max} = 1\times10^9$

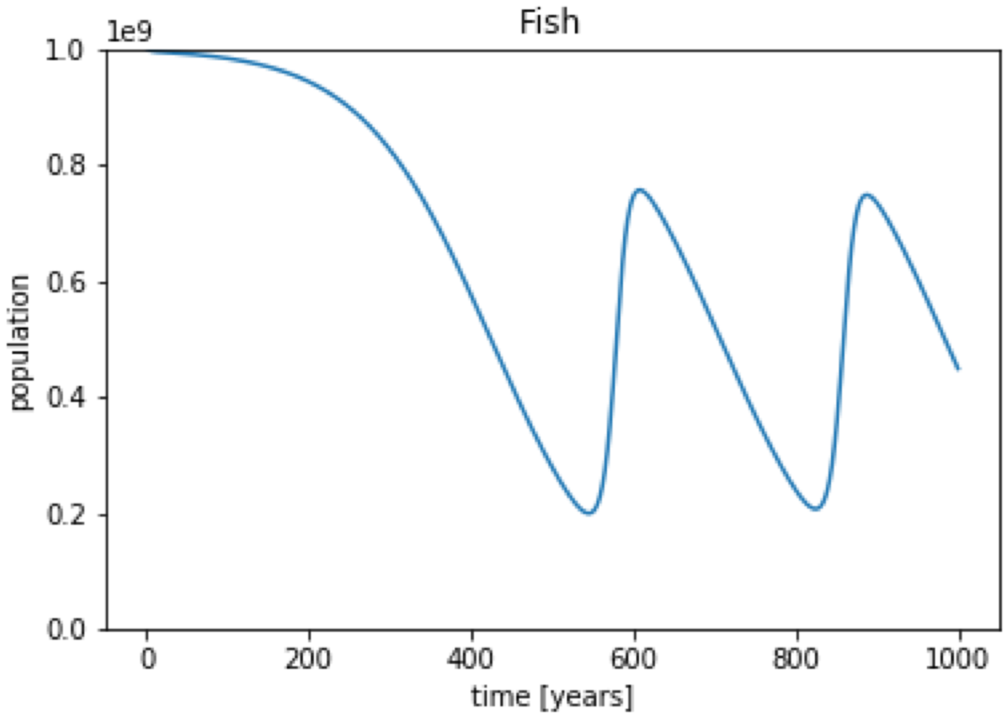
$R = 1000$

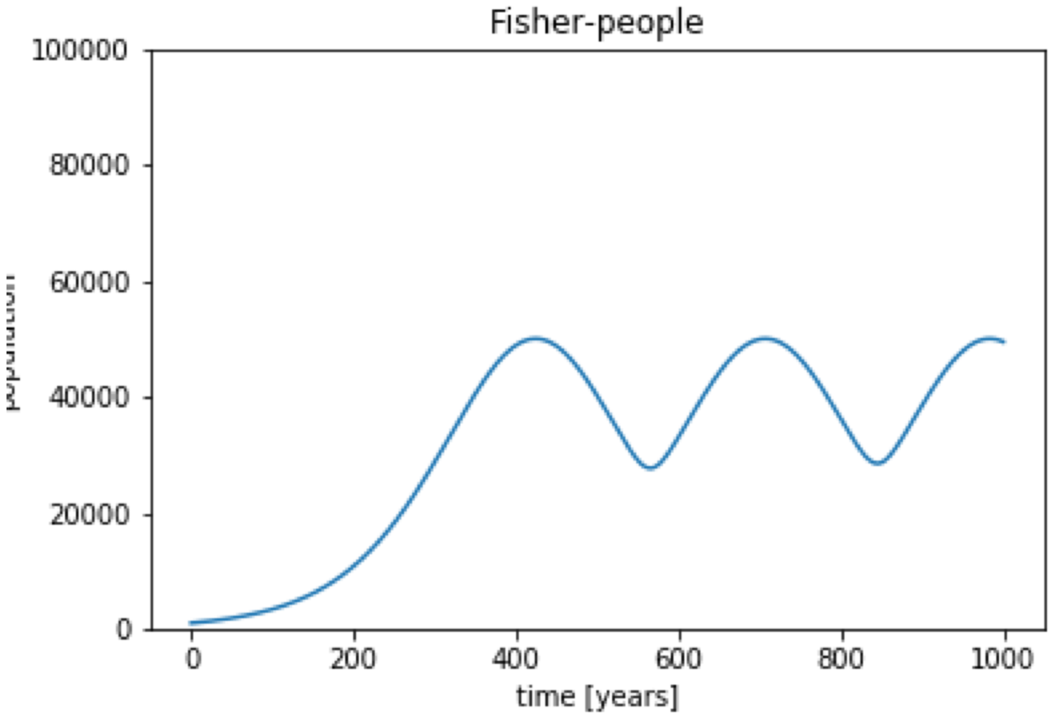
$P_{0,humans} = 1000$

$\tau_{humans} = 80 \text{ years}$

$c = 0.0001$

An oscillating relationship with nature





$P_{0,fish} = 1\times10^9$

$\tau_{fish} = 5 \text{ years}$

$P_{fish}^{max} = 1\times10^9$

$R = 1063$

$P_{0,humans} = 1000$

$\tau_{humans} = 80 \text{ years}$

$c = 0.0001$