Tunneling Effect

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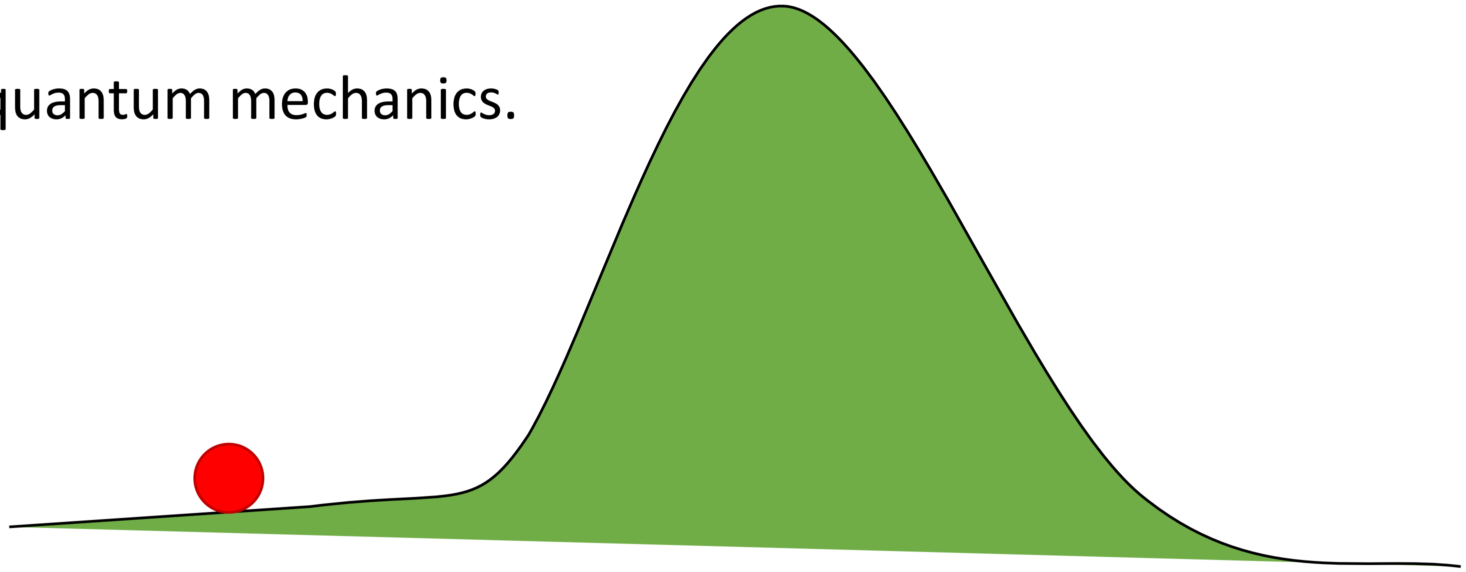
What is Tunneling?

- Consider a particle with an energy, E and a potential hill with potential, V. Classically, if E<V then the particle cannot overcome this barrier and will never roll to the other side. If E>V then the particle has enough energy to overcome the potential energy (V) at the top of the hill and will roll to the other side.

- This is not always true in quantum mechanics.

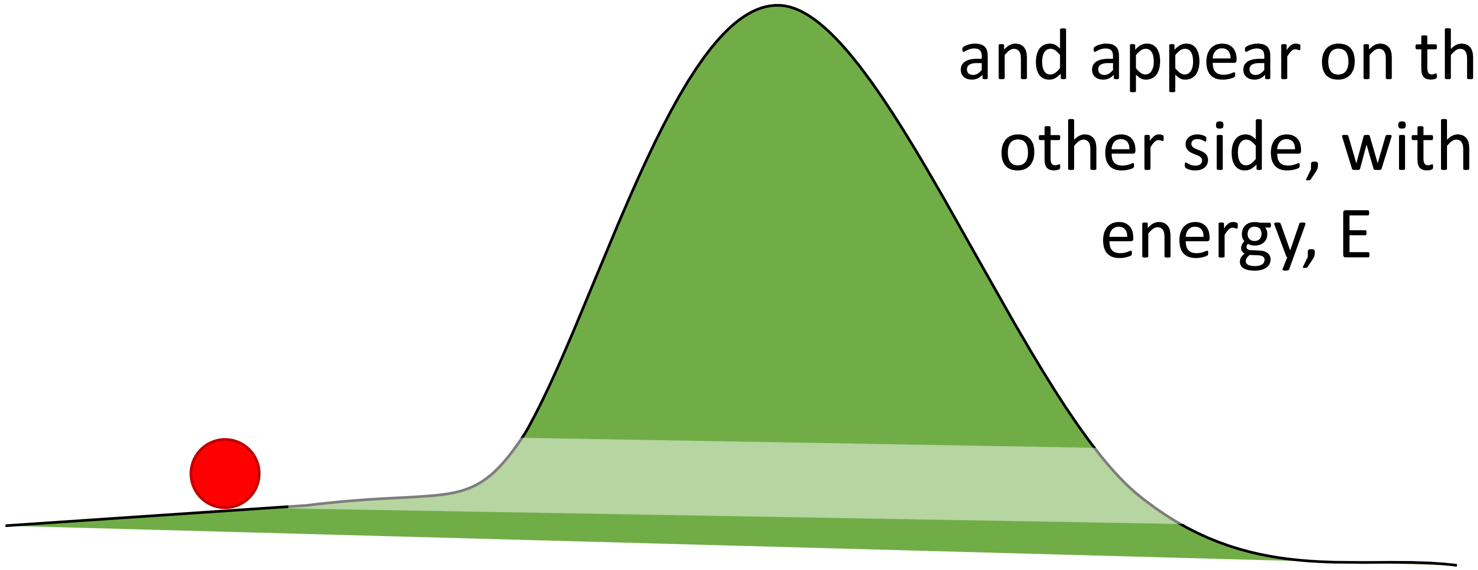
Classically, electrons must climb the potential hill to appear on the other side



What is Tunneling?

- In quantum mechanics, the particle can escape, despite its energy E being below the potential well, there is a probability of escape.

Quantum Mechanics allows an electron with less energy then required to overcome the potential, to tunnel through the barrier…



and appear on the other side, with energy, E

Using a continuous coupled pendulum to study wave dispersion and tunneling

- Obviously, this is something hard to grasp, and much harder to teach. Thus, experiments to demonstrate this concept is important for conceptual understanding.

- One way of doing this is through physical system which demonstrates wave motion with components we can control.

- So we will consider a coupled pendulum system.

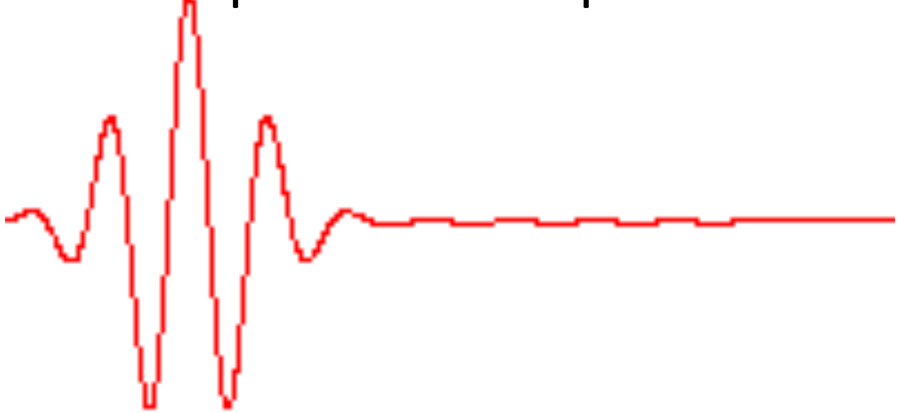
Dispersion

- Dispersion occurs when waves of different wavelengths have different propagation velocities.

- So, a wave packet of mixed wavelengths tends to spread out in space.

  - A wave packet is also referred to as an envelope of waves propagating.

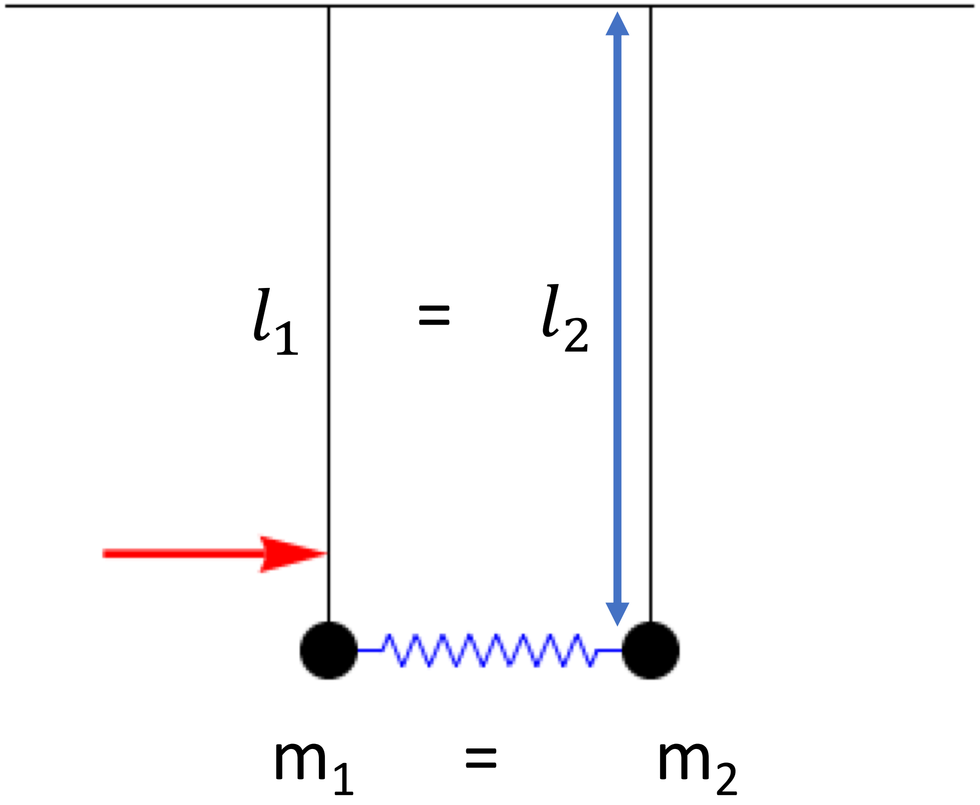
Dispersive wave packet



Non-dispersive wave packet

Finding a Dispersion Relation

Consider a coupled pendulum experiment



Newton's 2nd Law: For A Simple Pendulum

Mass 1

$$m\frac{d^2x_1}{dt^2} = m\ddot{x}_1 = -mg\sin\theta_1$$

Using the small angle approximation for large l:

$$m\ddot{x}_1 = -mg\frac{x_1}{l}$$

Mass 2

$$m\frac{d^2x_2}{dt^2} = m\ddot{x}_2 = -mg\sin\theta_2$$

By same approximation:

$$m\ddot{x}_2 = -mg\frac{x_2}{l}$$

Newtons 2nd Continued

Assuming our system obeys Hooke's law, the force exerted by the spring acts in the opposite direction of the displacement. This gives us the following equations of motion.

$$m\ddot{x}_1 = -mg\frac{x_1}{l} + \kappa(x_2 - x_1)$$
$$\therefore \quad \ddot{x}_1 = -\frac{g}{l}x_1 + \frac{\kappa}{m}(x_2 - x_1)$$

And

$$m\ddot{x}_2 = -mg\frac{x_2}{l} - \kappa(x_2 - x_1)$$
$$\therefore \quad \ddot{x}_2 = -\frac{g}{l}x_2 - \frac{\kappa}{m}(x_2 - x_1)$$

Newtons 2nd Continued

Combine the equations of motion to define the motion for pendula moving in identical phase with no relative change in position.

$$\ddot{x}_1 + \ddot{x}_2 = -\frac{g}{l}(x_1 + x_2) \qquad \rightarrow \qquad \ddot{x}_+ = -\frac{g}{l}(x_+)$$

By observation we see that we can write the solution as a cos function. Also note that

$$\sqrt{\frac{g}{l}} = \omega_p$$

$$\therefore \quad \ddot{x}_+ = \omega^2{}_p x_+$$

$$\therefore \quad x_+ = A_1 \cos\left(\omega_p t + \varphi_1\right)$$

Where A and φ are a set of initial or boundary conditions.

Newtons 2nd Continued

- Now, we combine the equations of motion to show the pendula separating or coming together.

$$\ddot{x}_1 - \ddot{x}_2 = (x_1 - x_2)\left(-\frac{g}{l} - \frac{2\kappa}{m}\right) \qquad \rightarrow \ddot{x}_- = -\left(\frac{g}{l} + \frac{2\kappa}{m}\right)x_-$$

By observation we see that $\omega = \sqrt{\frac{g}{l} + \frac{2k}{m}}$

And the solution to the above differential equation is similar to the previous solution for SHM.

General Solution

Our final general solution is

$$x = A_1 \cos(\omega_p t + \varphi_1) + A_2 \cos(\omega t + \varphi_2)$$

Dispersion Relation

Taking the previous result

$$\omega^2 = \frac{g}{l} + \frac{2\kappa}{m}$$

And recognizing that

$$\omega_p{}^2 = \frac{g}{l}$$

We can write that

$$\omega^2 = \omega_p{}^2 + 2\frac{\kappa}{m}$$

This is the angular frequency of a mass on a spring

Dispersion Equation Cont.

Using the wave speed relation

$$v = f\lambda$$

And

$$\omega = 2\pi f$$

And the wavenumber relation

$$k = \frac{2\pi}{\lambda}$$

We can rewrite our dispersion equation as

$$\omega^2 = {\omega_p}^2 + 2k^2v^2$$

Phase and Group Velocities

- Phase Velocity (v_p): The speed of a single sinusoidal traveling wave
- Group Velocity (v_g) : The velocity at which a whole envelope of waves propagate
- Observing the relation between these velocities and wave dispersion are essential in discussing tunneling.

Phase and Group Velocities

To derive, we can take two harmonic waves with close angular frequencies, k values, and of the same amplitude.

$$u(x,t) = Acos(\omega_1 t - k_1 x) + Acos(\omega_2 t - k_2 x)$$

Note: $\cos A + \cos B = 2cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

Which allows us to re-write as

$$\underbrace{2A\cos\left(\frac{\omega_2 - \omega_1}{2}t - \frac{k_2 - k_1}{2}x\right)}\cos\left(\frac{\omega_2 + \omega_1}{2}t - \frac{k_2 + k_1}{2}x\right)$$

This is the net amplitude

Phase Velocity

Take the second cosine in the previous summed wave equation

$$\cos\left(\frac{\omega_2+\omega_1}{2}t-\frac{k_2+k_1}{2}x\right)$$

To find the phase velocity, we want to find the condition such that $\bar{k}$x-$\bar{\omega}$t is constant with respect to time.

$$\bar{k}x-\bar{\omega}t = constant$$

$$\Rightarrow \boxed{\frac{dx}{dt}=\frac{\omega}{k}=v_p}$$

Group Velocity

We can obtain v_g by keeping the amplitude constant

$$\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) = constant \qquad \Rightarrow \qquad \left(\frac{\Delta\omega}{\Delta k}t - x\right) = constant$$

Now we can see the rate of propagation of our envelope as a function of time.

$$v_g = \frac{\omega_2 - \omega_1}{k_2 - k_1} = \frac{d\omega}{dk}$$

Phase and Group Velocity

$$v_p = \frac{\omega}{k}$$

$$v_g = \frac{\omega_2 - \omega_1}{k_2 - k_1} = \frac{d\omega}{dk}$$

Phase Velocity for Coupled Pendula

$$\omega^2 = \omega_p{}^2 + 2k^2v^2$$

$$\frac{\omega^2}{k^2} = \frac{\omega_p{}^2}{k^2} + 2v^2 = v_p{}^2$$

$$v_p = \sqrt{\frac{\omega_p{}^2}{k^2} + 2v^2}$$

And replacing with k we get

$$v_p = \frac{\sqrt{2} * v * \omega}{\sqrt{\omega^2 - \omega_p{}^2}}$$

When $\omega_p > \omega$, v_p becomes imaginary

Group Velocity for Continuous Pendula

$$v_g = \frac{d\omega}{dk}$$

$$v_g = \frac{d\omega}{dk}\left(\sqrt{{\omega_p}^2 + 2k^2v^2}\right) = \frac{2kv^2}{\omega} = \frac{2v^2}{v_p}$$

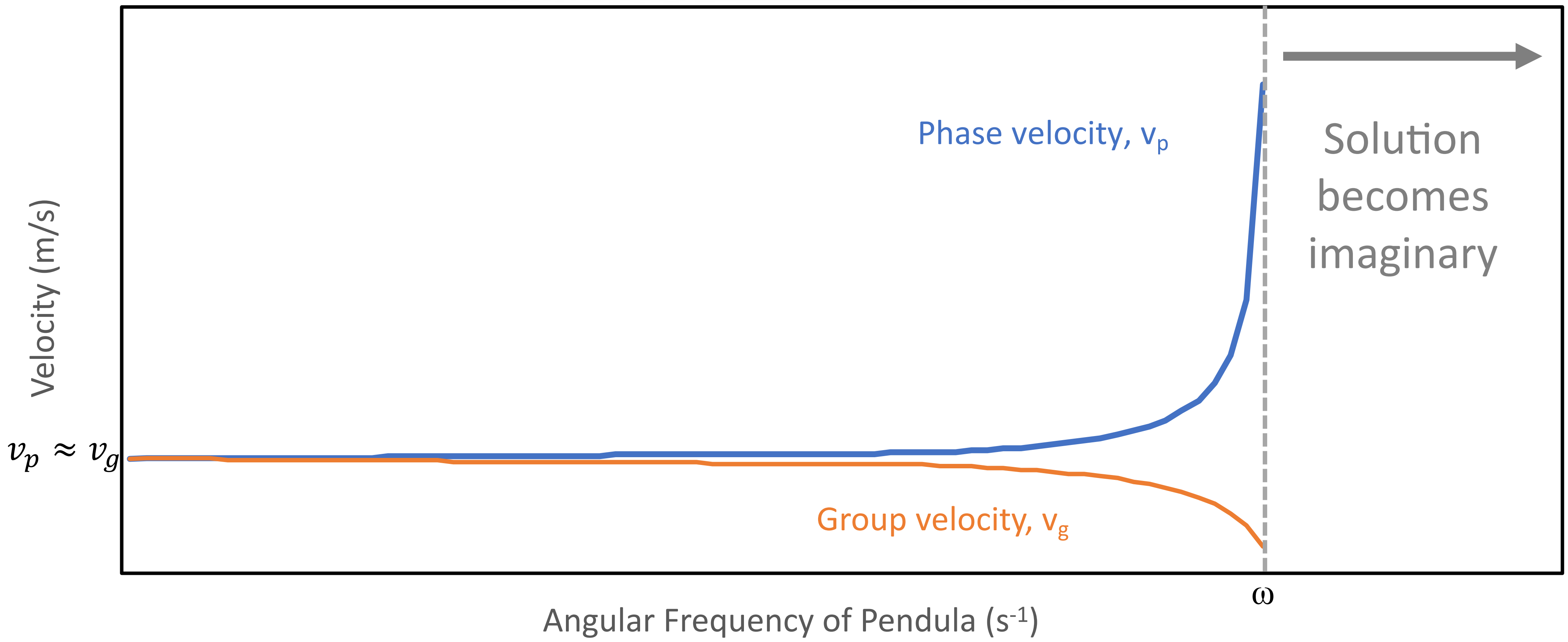
$$v_g = \frac{2v^2}{\sqrt{\frac{{\omega_p}^2}{k^2} + 2v^2}}$$

Now we can plug in k to get everything in terms of v, ω, and ω_p.

$$v_g = \frac{v\sqrt{2}}{\omega\sqrt{\frac{1}{\omega^2 - {\omega_p}^2}}}$$

When $\omega_p > \omega$, v_g becomes imaginary

Solutions for v_g and v_p



Imaginary Solution

- How to interpret an imaginary solution?

- Physically, the wave makes it through the region where $\omega < \omega_p$, much like the case of a particle tunneling through a potential barrier in quantum mechanics.

Coupled Pendulum System

- How to create this effect?
  - We can shorten the string length of a single pendula
- Increasing ω_p such that $\omega_p > \omega$ results imaginary solution

Importance of a model

- There are a lack of models and demonstrations to show tunneling effects.
- Mathematically, the result is contrary to the physical occurrence.
- Similar comparison to tunneling in quantum mechanics.

Experimentation

A spring-coupled pendulum system similar to what is described here has been build at SUNY Cortland

However, data has not been taken to demonstrate the tunneling phenomenon.