Analysis of optimal range of pendula length for a coupled-oscillator experiment

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In designing a system of coupled pendula-oscillators for the purpose of study wave dispersion, tunneling and other phenomena, it is important to establish as a design constraint the desired range of pendula lengths. While it is impossible to make the pendulum effect vanish completely, it is possible to make it arbitrarily small or large in comparison to the coupling force (spring force) between adjacent oscillators by making the pendula length longer or shorter, respectively. We take as our starting point for this analysis the dispersion relation presented in [1], namely

$$\omega^2 = \omega_p^2 + 4\omega_s^2 \sin^2\left(\frac{ka}{2}\right), \tag{1}$$

where $\omega_p^2 = g/l$ is the frequency associated with a free pendulum of length l, $\omega_s^2 = \kappa/m$ is the frequency associated with an elastic force (a spring of spring constant κ and mass m), k is the wavenumber of the oscillations and a is the equilibrium distance between oscillators. Note that in the limit of small values of ka (i.e. small distance between the oscillators or long-wavelength, equivalently), we may employ the small angle approximation for sine to get the more common dispersion relation for a continuous media with a cutoff,

$$\omega^2 \approx \omega_p^2 + \omega_s^2 (ka)^2 . \tag{2}$$

There are three independent strategies one might employ to enforce a change in cutoff frequency or wave tunneling effect:

- vary the external restoring force (pendula lengths)
- vary the inertia (masses)
- vary the coupling force (springs)

While all of these are viable solutions, including some combination thereof, it seems at first pass that it should be most effective to build a device capable of changing the ratio of external to internal force by changing the pendula length without having to swap any components (masses or springs), and therefore this offers the greatest flexibility for experimentation. In the following analysis we will focus only on the necessary range of pendula lengths to get the desired effects. To proceed, we will consider that motion will be driven by an external motor forcing a single oscillator. Thus, the frequency (ω) is fixed by the external driver. We wish to explore what range of pendula lengths will, in the limit of a long pendula, give the appearance of simply connected oscillators free of a gravitational restoring force, and in the limit of a short pendula, provide a gravitational force that dominates over the elastic forces between the oscillators.

Let us take as a rough design the inertial and coupling parameters used in [1], namely, $m \approx 0.2$ kg and $\kappa \approx 2$ N/m. For fixed frequency we should like to know first what upper limit to length we should have so that the external restoring force seems negligibly small. To do so we will consider the long-wavelength limit of Eq. 2 and take as a starting point that we want at least six oscialltors per wavelength so that we can observe a clear wave structure. More generally, let us consider $\lambda = n6a$, where we consider a reasonable range of $n = 1, 2, 3$. In this limit we have

$$\omega^2 = \omega_p^2 + \omega_s^2 \left(\frac{2\pi}{n6a} a \right)^2 \approx \omega_p^2 + \frac{1}{n^2}\omega_s^2. \tag{3}$$

In order to have the restoring force be negligible in comparison to the oscillator coupling, we should like $\omega_s^2/n^2 \gtrsim 10\, \omega_p^2$. Expressed as a solution for the length, l, we have

$$l_{upper} \gtrsim 10n^2 \frac{mg}{\kappa}. \tag{4}$$

Using the value of κ and m from [1] and taking $g \approx 10$ m/s^2, we have the condition $l_{upper} \gtrsim 10n^2$ meters. Clearly, this is not feasible and this informs us that we need to increase the elastic force by increasing the spring constant by at least a factor of 5 or decreasing the mass by a factor of 5. We want to avoid decreasing the mass because we want to continue to have the oscillator mass substantially larger than the mass of the springs themselves, which leads us to revise our choice of spring. An upper limit to the length of approximately 2 meters is feasible, and taking that as a constraint we inquire what spring constants provide that condition for the range of wavelength parameters. The following table summarizes these relationships.

n	λ	κ [N/m]
1	6a	10
2	12a	40
3	18a	90

This analysis suggests that we should look for springs with spring constants in the range of $10 - 90$ N/m, equivalent to roughly $0.06 - 0.54$ lbs/in, which is well within reason for a wide range of standard springs. In considering these systems, we should be mindful not only of the relative scale of these frequencies, but their absolute value as we want to make sure that these frequencies are reasonably attainable by our mechanical driver and measurable by a camera or other detection system. Let us consider $\omega^2 \approx \omega_s^2/n^2 = \kappa/mn^2$. Note that $\kappa/n^2 = 10$ N/m, hence, for all cases we have $\omega^2 \approx 50$ (rad/s)^2, or in terms of the frequency, we have $f_s \approx 1.1$ Hz (66 rpm), a nicely attainable value for a mechanical drive. As a sanity check, we find $f_p \approx 0.35$ Hz for a pendulum length of 2 meters, which is, as expected, about a factor of 3 lower than the frequency for the $n = 1$ case (recall that the frequencies entered quadratically in the dispersion relationship, and therefore also in our limit analysis).

We now inquire what minimum length we should need for the inverse condition to hold, that is, that the external restoring force should dominate over the internal elastic forces. The analysis on this part is confounded by the fact that as the pendula lengths are changed slowly over some distance the wave phase speed changes. Because the frequency is constant for the entire system, this implies that the wavelength must change. Returning to Eq. 2 and requiring that in the freely-propagating region we want a negligible restoring force, we have $\omega \approx \omega_s$. Expressing the wavelength explicitly in the dispersion relation, we have

$$\omega_s^2 \approx \omega_p^2 + \omega_s^2 \left(\frac{2\pi}{\lambda} a\right)^2. \tag{5}$$

Or, solving for λ we have

$$\lambda^2 = \frac{(2\pi a)^2}{1 - \frac{\omega_p^2}{\omega_s^2}}. \tag{6}$$

With a goal of developing a system where the wavelength not only changes, but becomes imaginary (representing an evanescent wave, or tunneling through a forbidden region), we want the RHS of Eq. 6 to be negative. Equivalently, we require that $\omega_p^2 > \omega_s^2$. This provides our desired condition for the lower limit of the length, namely,

$$l_{lower} \gtrsim \frac{mg}{\kappa}. \tag{7}$$

Using the values derived earlier we find l_{lower} values of 0.2 m, 0.05m and approximately, 0.02m, corresponding to $n = 1, 2, 3$, respectively. Note that the $n = 2$ and $n = 3$ cases seem to be too small, especially considering that the lower limit is the critical condition for evanescence and that to get a truly evanescent wave structure we might like $\omega_p^2 \approx 2\omega_s^2$, which requires halving all these lower limit distances. Under this stricter condition, examining the $n = 1$ solution for $l = 0.10$ m, we would have an evanescent scale length (an e-folding distance) of $2\pi a$, which seems entirely appropriate for exploring tunneling phenomena. Note as well that by reducing the drive frequency by a factor of 2 the critical pendulum length and the pendulum length for a $2\pi a$ e-folding scale are doubled.

Therefore, so as the lower limit be not too small, and the upper limit too large, it seems that we must accept $n \approx 1$ and consequently $k/m \approx 50$ (rad/s)^2. Taking $m = 0.2$ kg then imposes the condition that we must use a spring with a spring constant of approximately 10 N/m, equivalent to about 0.06 lbs/in. This imposes a long pendulum limit of approximately 2 meters, and a short pendulum length of 0.1 m to produce an e-folding evanescent scale length equal to $2\pi a$.

[1] Liang *et al.*, AJP **83**, 389 (2015).