Lagrange Points and Regionally Conserved Quantities

Supplementary Notes

These supplementary notes contain additional details of the mathematical analysis that support the main text. Three main topics are presented here: (1) an analysis of the solutions for the L_1 and L_2 points, (2) an analysis of the solution for the L_3 point, (3) a comparison of two dimensionless forms, and (4) a comparison of the contour plots that result from Eqs. [6] and [7] of the main text.

I. COEFFICIENTS OF THE SERIES SOLUTIONS FOR L_1 AND L_2

The defining condition of case A is $\theta = 0$, which results in the following form for the dimensionless force equation for equilibrium, which is the same as Eq. [17] in the main text),

$$-\frac{1}{(x+\mu)^2} \mp \frac{\mu}{(x-1)^2} + \frac{x}{(1+\mu)^2} = 0. \tag{1}$$

Cross multiplying the denominators produces a numerator with the same zeros as above, which is given the identifier $f(x;\mu) = -(1+\mu)^2(x-1)^2 \mp \mu(1+\mu)^2(x+\mu)^2 + x(x+\mu)^2(x-1)^2$. Expanding all terms in $f(x;\mu)$ and ordering them by powers of x yields

$$\begin{aligned} f(x;\mu) = {} & \left[-1 - 2\mu - \mu^2 \mp \mu^3 \mp 2\mu^4 \mp \mu^5\right] x^0 + \left[2 + 4\mu + (3 \mp 2)\mu^2 \mp 4\mu^3 \mp 2\mu^4\right] x^1 \\ & + \left[-1 \mp \mu + (-3 \mp 2)\mu^2 \mp \mu^3\right] x^2 + \left[1 - 4\mu + \mu^2\right] x^3 + \left[-2 + 2\mu\right] x^4 + x^5. \end{aligned} \tag{2}$$

The prior equation can be efficiently represented in matrix form as

$$f(x;\mu) = \begin{pmatrix} 1 \\ \mu \\ \mu^2 \\ \mu^3 \\ \mu^4 \\ \mu^5 \end{pmatrix}^T \begin{pmatrix} -1 & 2 & -1 & 1 & -2 & 1 \\ -2 & 4 & \mp 1 & -4 & 2 & 0 \\ -1 & 3 \mp 2 & -3 \mp 2 & 1 & 0 & 0 \\ \mp 1 & \mp 4 & \mp 1 & 0 & 0 & 0 \\ \mp 2 & \mp 2 & 0 & 0 & 0 & 0 \\ \mp 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix}. \tag{3}$$

A transformation from the variable x to Δ is defined by $x = 1 + \Delta$. The associated matrix defining this transformation for the x^0 through x^5 is

$$\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \Delta \\ \Delta^2 \\ \Delta^3 \\ \Delta^4 \\ \Delta^5 \end{pmatrix}. \tag{4}$$

Multiplying the matrices of Eqs. 3 and 4 produces

$$f(\Delta;\mu) = \begin{pmatrix} 1 \\ \mu \\ \mu^2 \\ \mu^3 \\ \mu^4 \\ \mu^5 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 3 & 3 & 1 \\ \mp 1 & \mp 2 & \mp 1 & 4 & 2 & 0 \\ \mp 4 & \mp 6 & \mp 2 & 1 & 0 & 0 \\ \mp 6 & \mp 6 & \mp 1 & 0 & 0 & 0 \\ \mp 4 & \mp 2 & 0 & 0 & 0 & 0 \\ \mp 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \Delta \\ \Delta^2 \\ \Delta^3 \\ \Delta^4 \\ \Delta^5 \end{pmatrix}. \tag{5}$$

Completing the vector multiplication of these terms and applying some further algebraic simplifications leads to the compact form of $f(\Delta;\mu)$ represented as Eq. 20 in the main text,

$$f(\Delta;\mu) = \mp\mu(1+\mu)^2(1+\mu+\Delta)^2 + \Delta^3\left[\left(3+4\mu+\mu^2\right)+(3+2\mu)\Delta+\Delta^2\right]. \tag{6}$$

This equation can be solved by expanding Δ in powers of $\mu^{1/3}$ or, since the fraction $1/3$ occurs throughout this analysis, a better choice is $(\mu/3)^{1/3}$. A final transformation using $\mu = 3z^3$ yields a form that allows for an expansion in integer powers of z, where the coefficients of the terms are all rational fractions. The transformation matrix from μ to z is

$$\begin{pmatrix} 1 \\ \mu \\ \mu^2 \\ \mu^3 \\ \mu^4 \\ \mu^5 \end{pmatrix}^T = \begin{pmatrix} 1 \\ z^3 \\ z^6 \\ z^9 \\ z^{12} \\ z^{15} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 27 & 0 & 0 \\ 0 & 0 & 0 & 0 & 81 & 0 \\ 0 & 0 & 0 & 0 & 0 & 243 \end{pmatrix}. \tag{7}$$

Multiplying the prior matrix into the form of $f(\Delta;\mu)$ gives $f(\Delta;z)$ as

$$f(\Delta;z) = \begin{pmatrix} 1 \\ z^3 \\ z^6 \\ z^9 \\ z^{12} \\ z^{15} \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 3 & 3 & 1 \\ \mp 3 & \mp 6 & \mp 3 & 12 & 6 & 0 \\ \mp 36 & \mp 54 & \mp 18 & 9 & 0 & 0 \\ \mp 162 & \mp 162 & \mp 27 & 0 & 0 & 0 \\ \mp 324 & \mp 162 & 0 & 0 & 0 & 0 \\ \mp 243 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \Delta \\ \Delta^2 \\ \Delta^3 \\ \Delta^4 \\ \Delta^5 \end{pmatrix}. \tag{8}$$

While the matrix form is convenient for calculating transformations such as these, analysis of the actual solutions is better done by unpacking the matrix and representing the polynomial in its long form. This above matrix form is equivalent to

$$\begin{aligned} f(\Delta;z) = &\mp \left[3z^3 + 36z^6 + 162z^9 + 324z^{12} + 243z^{15}\right]\Delta^0 \\ &\mp \left[6z^3 + 54z^6 + 162z^9 + 162z^{12}\right]\Delta^1 \mp \left[3z^3 + 18z^6 + 27z^9\right]\Delta^2 \\ &+ \left[3z^0 + 12z^3 + 9z^6\right]\Delta^3 + \left[3z^0 + 6z^3\right]\Delta^4 + \Delta^5. \end{aligned} \tag{9}$$

We are now in a good position to calculate the series representation using $\Delta = \sum_{n=1}^{\infty} a_n z^n$, from which Δ as a function of μ can be recovered using $z = (\mu/3)^{1/3}$. The a_n are calculated by ordering Eq. 9 by powers of z. The solution for the unknown a_n begins by solving for a_1

at order z^3 by taking the ∓ 3 from the z^3 coefficient of the Δ^0 term, and the 3 from the z^0 coefficient of the Δ^3 term, which gives the following.

z^3:

$$\mp 3 + 3a_1^3 = 0 \tag{10}$$

$$a_1 = \pm 1 \tag{11}$$

For all subsequent orders, the highest order coefficient (the unknown for which we are solving) always enters as $9a_1^2 a_n$ through the Δ^3 term at order z^{n+2} (e.g. the a_2 term appears as the highest coefficient in the equation for z^4). Given that $a_1^2 = 1$, it follows that a_n will equal all other terms divided by 9. At order z^4, the Δ^1 term enters through its z^3 coefficient, cross terms in Δ^3 enter as $3a_1^2 a_2$, and Δ^4 terms enter as a_1^4.

z^4:

$$\mp 6a_1 + 3\left(3a_1^2 a_2\right) + 3a_1^4 = 0 \tag{12}$$

$$a_2 = \frac{\pm 6a_1 - 3}{9} = \frac{1}{3} \tag{13}$$

As we move to higher order, the number of terms expands due to the increasing number of ways that cross terms can generate the appropriate power of z.

z^5:

$$\mp 6a_2 \mp 3a_1^2 + 3\left(3a_1 a_2^2 + 3a_1^2 a_3\right) + 3\left(4a_1^3 a_2\right) + a_1^5 = 0 \tag{14}$$

$$a_3 = \mp\frac{1}{9} \tag{15}$$

The z^6 order sees the entrance of the next term from the coefficient of Δ^0, which has the value 36. This is a symmetry-breaking term that causes the magnitudes of the a_n for the L_1 and L_2 branches to depart from each other.

z^6:

$$\mp 36 \mp 6a_3 \mp 3\left(2a_1a_2\right) + 3\left(a_2^3 + 6a_1a_2a_3 + 3a_1^2a_4\right) + 12a_1^3 + 3\left(6a_1^2a_2^2 + 4a_1^3a_3\right) + 5a_1^4a_2 = 0 \quad (16)$$

$$a_4 = \begin{cases} +\frac{212}{81} & \mathrm{L}_2(+\text{branch}) \\ -\frac{220}{81} & \mathrm{L}_1(-\text{branch}) \end{cases} \quad (17)$$

z^7:

$$\mp 6\left(a_4\right) \mp 54\left(a_1\right) \mp 3\left(a_2^2 + 2a_1a_3\right) + 3\left(3a_1a_3^2 + 3a_2^2a_3 + 6a_1a_2a_4 + 3a_1^2a_5\right)$$
$$+ 12\left(3a_1^2a_2\right) + 3\left(4a_1a_2^3 + 12a_1^2a_2a_3 + 4a_1^3a_4\right) + 6\left(a_1^4\right) + \left(10a_1^3a_2^2 + 5a_1^4a_3\right) = 0 \quad (18)$$

$$a_5 = \begin{cases} +\frac{124}{243} & \mathrm{L}_2(+\text{branch}) \\ +\frac{92}{243} & \mathrm{L}_1(-\text{branch}) \end{cases} \quad (19)$$

z^8:

$$\mp 6\left(a_5\right) \mp 54\left(a_2\right) \mp 3\left(2a_2a_3 + 2a_1a_4\right) \mp 18\left(a_1^2\right) + 3\left(3a_2a_3^2 + 3a_2^2a_4 + 6a_1a_3a_4 + 6a_1a_2a_5 + 3a_1^2a_6\right)$$
$$+ 12\left(3a_1a_2^2 + 3a_1^2a_3\right) + 3\left(a_2^4 + 6a_1^2a_3^2 + 12a_1a_2^2a_3 + 12a_1^2a_2a_4 + 4a_1^3a_5\right) + 6\left(4a_1^3a_2\right)$$
$$+ \left(10a_1^2a_2^3 + 20a_1^3a_2a_3 + 5a_1^4a_4\right) = 0 \quad (20)$$

$$a_6 = \mp\frac{4}{9} \quad (21)$$

Interestingly, symmetry is briefly re-established at the 6^{th}-order, but is broken again for subsequent terms.

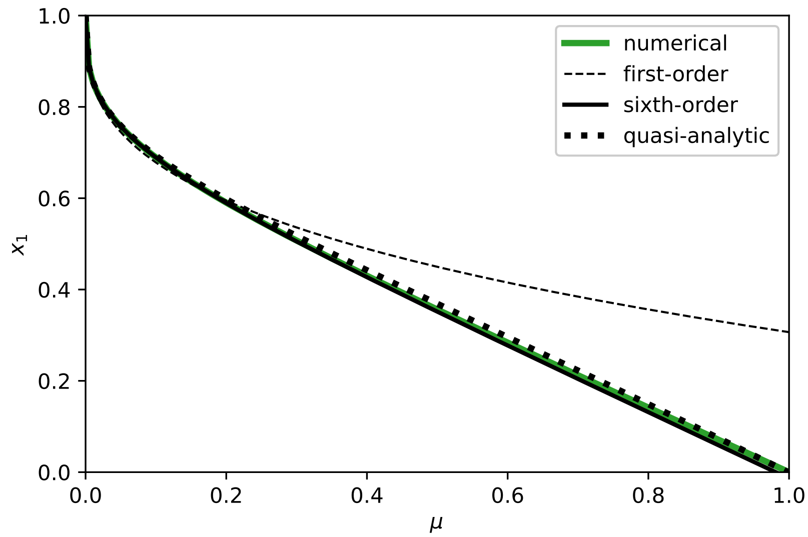

FIG. 1. Four solutions for the L_1 points are shown: the numerical solution (solid green), the first-order approximation (thin black dashed), the 6^{th}-order approximation (medium black solid), and the 4^{th}-order quasi-analytic approximation described in Eq. [23] of the main text.

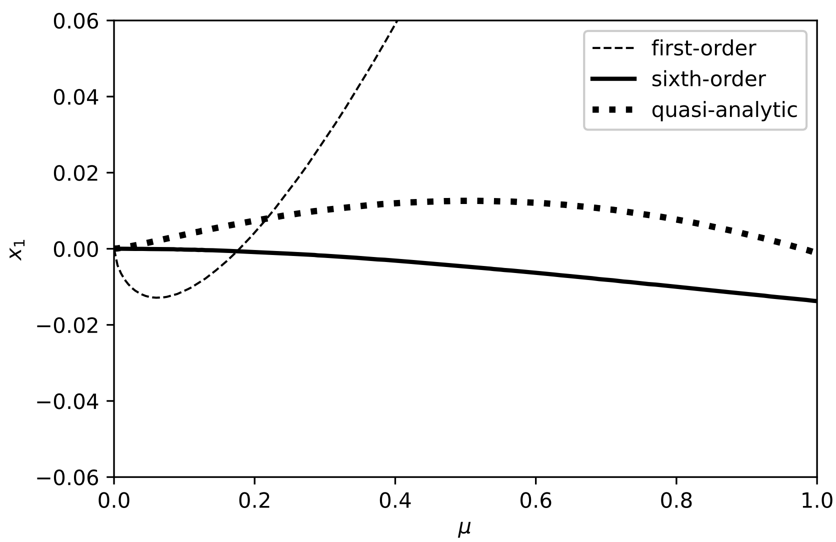

FIG. 2. The difference of each approximate solution for the location of L_1 from the numerical solution.

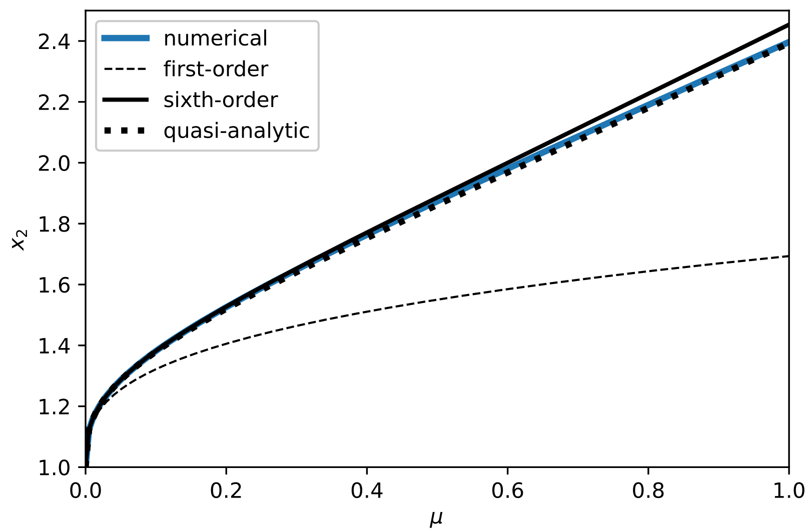

FIG. 3. Four solutions for the L_1 points are shown: the numerical solution (solid blue), the first-order approximation (thin black dashed), the 6^{th}-order approximation (medium black solid), and the 4^{th}-order quasi-analytic approximation described in Eq. [24] of the main text.

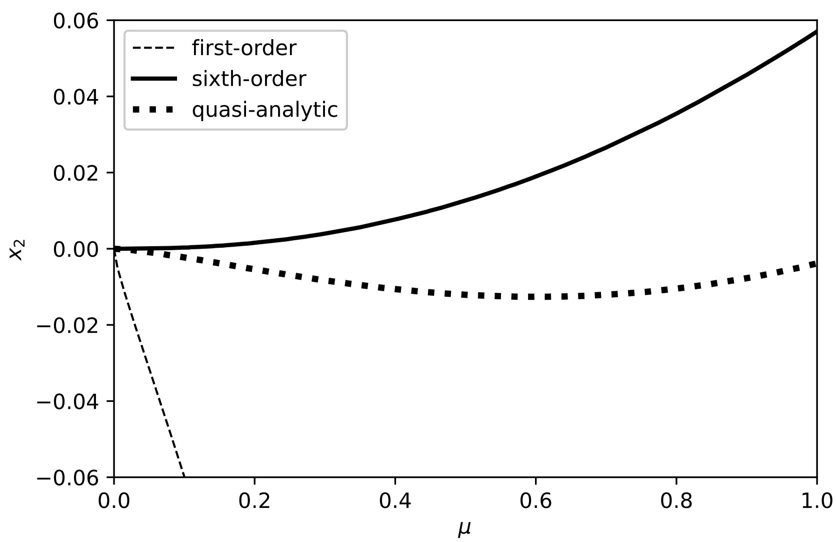

FIG. 4. The difference of each approximate solution and the numerically calculated solutions.

II. COEFFICIENTS OF THE SERIES SOLUTION FOR L_3

The defining condition of case B is $\theta = \pi$, which results in $\cos(\theta) = -1$, $x_1' = \pm(x-\mu)$, depending on whether x is greater than μ (upper sign) or less than μ (lower sign), and $x_2' = x+1$. As noted in the text, there are no solutions for $x < \mu$, so only one branch needs to be explored. However, in the interest of providing a complete solution so that others may verify this analysis, both forms are kept here. The physically allowable branch (describing the L_3 point) will be represented by the upper sign in this section, and the non-physical branch will be represented by the lower sign. The governing polynomial particular to case B, which is derived from Eq. [14] of the main text, is $g(x;\mu) = \mp(1+\mu)^2(x-1)^2 - \mu(1+\mu)^2(x+\mu)^2 + x(x+\mu)^2(x-1)^2$. Grouping by powers of x, this is equivalent to

$$\begin{aligned} g(x;\mu) = {} & \left[\mp 1 \mp 2\mu \mp \mu^2 - \mu^3 - 2\mu^4 - \mu^5\right] x^0 + \left[\mp 2 \mp 4\mu + (3\mp 2)\mu^2 + 4\mu^3 + 2\mu^4\right] x^1 \\ & \left[\mp 1 + (-3\mp 2)\mu \mp \mu^2 - \mu^3\right] x^2 + \left[1 - 4\mu + \mu^2\right] x^3 + \left[2 - 2\mu\right] x^4 + x^5. \end{aligned} \tag{22}$$

The prior can be presented in matrix form as

$$g(x;\mu) = \begin{pmatrix} 1 \\ \mu \\ \mu^2 \\ \mu^3 \\ \mu^4 \\ \mu^5 \end{pmatrix}^T \begin{pmatrix} \mp 1 & \mp 2 & \mp 1 & 1 & 2 & 1 \\ \mp 2 & \mp 4 & -3\mp 2 & -4 & -2 & 0 \\ \mp 1 & 3\mp 2 & \mp 1 & 1 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix}. \tag{23}$$

As with the prior case, the transformation from x to $1+\Delta$ is accomplished by multiplying the prior matrix by Eq. 4 on the right,

$$g(\Delta;\mu) = \begin{pmatrix} 1 \\ \mu \\ \mu^2 \\ \mu^3 \\ \mu^4 \\ \mu^5 \end{pmatrix}^T \begin{pmatrix} 4\mp 4 & 16\mp 4 & 25\mp 1 & 19 & 7 & 1 \\ -9\mp 8 & -26\mp 8 & -27\mp 2 & -12 & -2 & 0 \\ 4\mp 4 & 6\mp 4 & 3\mp 1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \Delta \\ \Delta^2 \\ \Delta^3 \\ \Delta^4 \\ \Delta^5 \end{pmatrix}. \tag{24}$$

This last form does not possess the interesting symmetry properties of the L_1 and L_2 branches since the point of symmetry in these branches occurs at $x = \mu$, not $x = 1$. Given that the only physical solutions come from the branch for the upper sign, the lower sign is hereafter abandoned. The matrix particular to the L_3 branch is

$$g(\Delta;\mu) = \begin{pmatrix} 1 \\ \mu \\ \mu^2 \\ \mu^3 \\ \mu^4 \\ \mu^5 \end{pmatrix}^T \begin{pmatrix} 0 & 12 & 24 & 19 & 7 & 1 \\ -17 & -34 & -29 & -12 & -2 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \Delta \\ \Delta^2 \\ \Delta^3 \\ \Delta^4 \\ \Delta^5 \end{pmatrix}. \tag{25}$$

Further analysis may be aided by recasting the matrix formulation in a more traditional format as

$$\begin{aligned} g(\Delta;\mu) = {} & \left[-17\mu + 2\mu^3 - \mu^5\right]\Delta^0 + \left[12 - 34\mu + 2\mu^2 + 2\mu^3 + 2\mu^4\right]\Delta^1 \\ & + \left[24 - 29\mu + 2\mu^2 - \mu^3\right]\Delta^2 + \left[19 - 12\mu + \mu^2\right]\Delta^3 + \left[7 - 2\mu\right]\Delta^4 + \Delta^5. \end{aligned} \tag{26}$$

Similar to the prior analysis, a series expansion is used to represent $\Delta(x;\mu)$. However, unlike the prior solution, this expansion involves only integer powers of μ and can be expressed as $\sum_{n=1}^{\infty} a_n \mu^n$. Collecting terms by powers of μ yields the following series of equations that determine the coefficients of this expansion.

μ^1:

$$-17 + 12a_1 = 0 \tag{27}$$

$$a_1 = +\frac{17}{12} \tag{28}$$

μ^2:

$$12a_2 - 34a_1 + 24a_1^2 = 0 \tag{29}$$

$$a_2 = 0 \tag{30}$$

Given that a_2 is identically zero, it is dropped in all subsequent equations.

μ^3:

$$2 + 12a_3 + 2a_1 - 29a_1^2 + 19a_1^3 \tag{31}$$

$$a_3 = -\frac{1127}{12^4} \tag{32}$$

μ^4:

$$12a_4 - 34a_3 + 2a_1 + 48\left(2a_1a_3\right) + 2a_1^2 - 12a_1^3 + 7a_1^4 = 0 \tag{33}$$

$$a_4 = +\frac{19\ 159}{12^5} \tag{34}$$

μ^5:

$$-1 + 12a_5 - 34a_4 + 2a_3 + 2a_1 + 24\left(2a_1a_4\right) - 29\left(2a_1a_3\right) - a_1^2 + 19\left(3a_1^2a_3\right) + a_1^3 - 2a_1^4 + a_1^5 = 0 \tag{35}$$

$$a_5 = -\frac{3\ 217\ 389}{12^7} \tag{36}$$

μ^6:

$$\begin{aligned} 12a_6 - 34a_5 + 2a_4 + 2a_3 + 24\left(a_3^2 + 2a_1a_5\right) - 29\left(2a_1a_4\right) + 2\left(2a_1a_3\right) \\ + 19\left(3a_1^2a_4\right) - 12\left(3a_1^2a_3\right) + 7\left(4a_1^3a_3\right) = 0 \end{aligned} \tag{37}$$

$$a_6 = +\frac{145\ 523\ 287}{12^8} \tag{38}$$

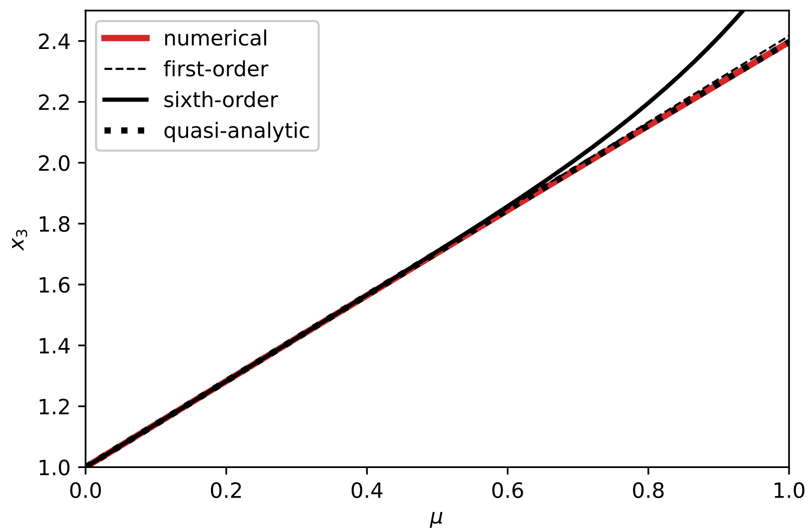

FIG. 5. Four solutions for the L_1 points are shown: the numerical solution (solid red), the first-order approximation (thin black dashed), the 6^{th}-order approximation (medium black solid), and the 3^{rd}-order quasi-analytic approximation described in Eq. [26] of the main text.

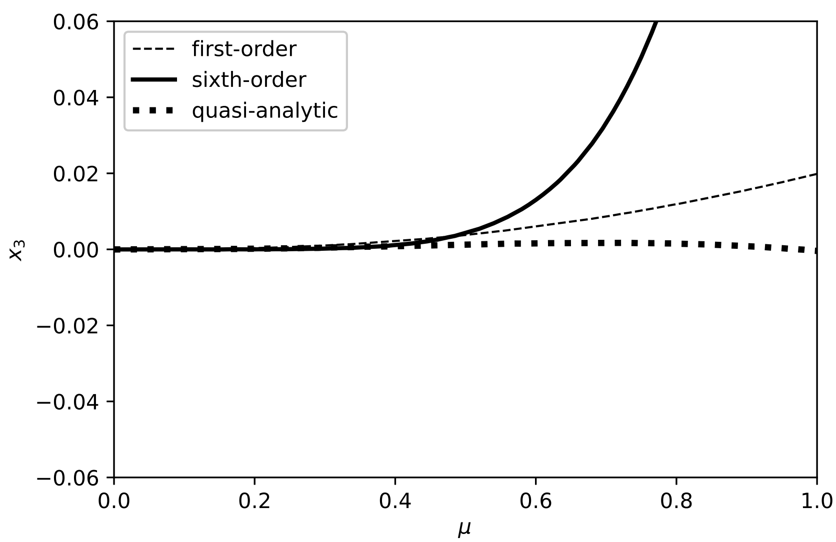

FIG. 6. The difference of each approximate solution and the numerically derived solutions is presented.

III. COMPARISON OF DIMENSIONLESS FORMS

The normalizing mass scale in this work was taken to be m_1. This choice implied that the lengths should be scaled by r_2, a result of the center of mass condition, $m_1 r_1 = m_2 r_2$, which is equivalent to $m_2/m_1 = r_1/r_2$. This normalization scheme will be compared here to that used in Refs. [4], [5], and [6], which are normalized using $R = r_1 + r_2$ and $M = m_1 + m_2$. This leads to a different definition of the mass parameter, $\mu^* = m_2/(m_1 + m_2)$, and, consequently, different representations of the series expansions. The goal of this section is to establish a connection between these representations to show that they give identical results.

To relate the two forms and show that they are, fundamentally, accurate representations of the solutions, two things are needed: a relation between μ and μ^*, and a relation between the x representations of this work and the x^* representations of Ref. [4] (note: Refs. [5] and [6] provide first-order approximations, only Ref. [4] provides expansions to higher order). The two forms of the mass ratio are related by $\mu = \mu^*/(1-\mu^*)$ and its inverse, $\mu^* = \mu/(1+\mu)$. The conversion between the representations is completed by noting that these forms must be equal when scaled to dimensional values. That is, it must be that $r = r_2 x = (r_1 + r_2)\, x^*$, from which it follows that x^* can be generated from the series presented here through the relation $x^* = r_2/\,(r_1 + r_2)\, x$. This is equivalent to $x^* = x/\,(1 + \mu) = x\,(1 - \mu^*)$.

A. Comparison of Representations for the L_1 and L_2 Points

The reader should note that Ref. [4] uses an older naming convention where the names of the L_1 and L_2 are swapped compared to the contemporary standard; the prior discussion has relabeled the L_1 and L_2 expressions from Ref. [4] using the current convention.

Solutions calculated using the two different normalization methods described in this work are presented in Figs. 7 and 8. For the sake of brevity, the r_2 and m_1 normalization used in this work will be referred to as the μ normalization, and the $r_1 + r_2$ and $m_1 + m_2$ normalization used in Refs. [4], [5], and [6] will be referred to as the μ^* normalization. There are many similarities between the forms presented in Figs. 7 and 8, and also some significant differences. All curves using the μ normalization are monotonic, whereas there is a local maximum near $\mu^* = 0.179$ for the L_2 curve in the μ^* normalization. The L_3 points increase in radius under the μ normalization, and decrease under the μ^* normalization,

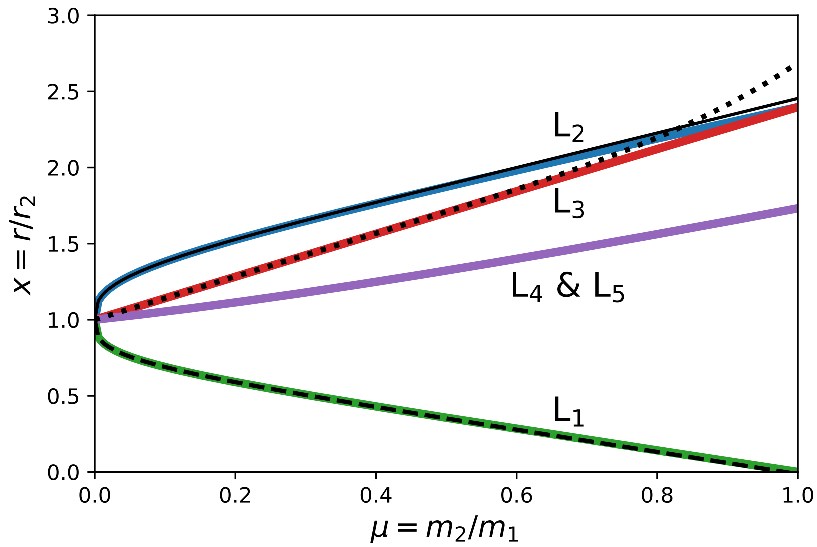

FIG. 7. Representations of solutions using the r_2 and m_1 normalization used in this work, that is, $\mu = m_2/m_1$. The solid color curves are the numerical or exact solutions. The black lines are the the 6^{th}-order truncated series representations for L_1 (dashed), L_2 (solid), and L_3 (dotted).

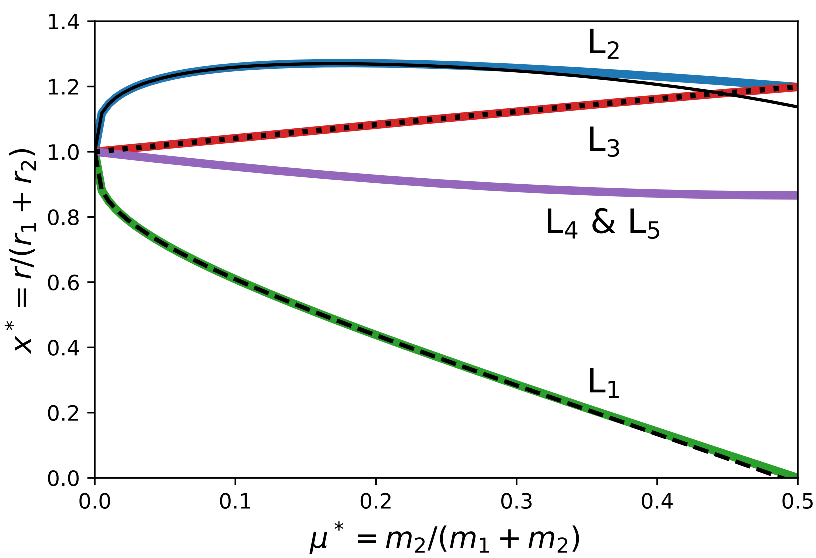

FIG. 8. Similar to Fig. 7, except using the $\mu^* = m_2/(m_1 + m_2)$ normalization.

which is to say that its radius grows more slowly than $r_1 + r_2$. Also interesting is that the L_2 solution is more accurate in the μ representation, whereas the L_3 representation is more accurate in the μ^* representation.

In presenting the series expansions for the L_1 and L_2 points, Ref. [4] uses an expansion in a parameter ν defined as $\nu = (\mu^*/3(1-\mu^*))^{1/3}$. Interestingly, and important for this analysis, this is equivalent to $\nu = (\mu/3)^{1/3}$. Therefore, in converting from $x_{1,2}$ to $x^*_{1,2}$, the μ's in $x_{1,2}$ can be left as they are, and it is only necessary to divide $x_{1,2}$ by $1+\mu$. The solutions for L_1 and L_2 are generically represented as $x_j = 1 + \sum_{n=1}^{\infty} a_n (\mu/3)^{n/3}$, where j is either 1 or 2. To generate the infinite series expansion of x^* from x, it is better to convert polynomial division into polynomial multiplication using the geometric series formula, $1/(1+\mu) = 1 - \mu + \mu^2 + \mathcal{O}(\mu^3)$. Since the series expansions for L_1 and L_2 proceed in powers of $(\mu/3)^{1/3}$, the conversion is better expressed as $x^* = 1/(1+\mu) + (x-1)\left[1 - 3(\mu/3)^{3/3} + 9(\mu/3)^{6/3} + \mathcal{O}(\mu^3)\right]$. Somewhat strangely, the expansions presented in Ref. [4] begin with $1-\mu^*$, and then use ν for all terms thereafter. It is therefore convenient that the leading factor of $1/(1+\mu)$ in the conversion is equal to $1-\mu^*$. Proceeding thusly, the formula representing the conversion from x to x^* out to 6^{th}-order is

$$
\begin{aligned}
x_j^* \approx 1 - \mu^* + a_1\left(\frac{\mu}{3}\right)^{1/3} + a_2\left(\frac{\mu}{3}\right)^{2/3} + a_3\left(\frac{\mu}{3}\right)^{3/3} \\
+ [a_4 - 3a_1]\left(\frac{\mu}{3}\right)^{4/3} + [a_5 - 3a_2]\left(\frac{\mu}{3}\right)^{5/3} + [a_6 - 3a_3]\left(\frac{\mu}{3}\right)^{6/3}.
\end{aligned} \tag{39}
$$

The series expansions for the L_1 and L_2 points with coefficients defined out to 6^{th}-order, using the μ representation of this work, are

$$
x_1 = 1 - \left(\frac{\mu}{3}\right)^{1/3} + \frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3} + \frac{1}{9}\left(\frac{\mu}{3}\right)^{3/3} - \frac{220}{81}\left(\frac{\mu}{3}\right)^{4/3} + \frac{92}{243}\left(\frac{\mu}{3}\right)^{5/3} + \frac{4}{9}\left(\frac{\mu}{3}\right)^{6/3} + \mathcal{O}(\mu^{7/3}) \tag{40}
$$

and

$$
x_2 = 1 + \left(\frac{\mu}{3}\right)^{1/3} + \frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3} - \frac{1}{9}\left(\frac{\mu}{3}\right)^{3/3} + \frac{212}{81}\left(\frac{\mu}{3}\right)^{4/3} + \frac{124}{243}\left(\frac{\mu}{3}\right)^{5/3} - \frac{4}{9}\left(\frac{\mu}{3}\right)^{6/3} + \mathcal{O}(\mu^{7/3}). \tag{41}
$$

Applying the equations for the coefficients defined in Eq. 39 to Eqs. 40 and 41 yields the representations of x_j^* in terms of μ, which are

$$x_1^* = 1 - \mu^* - \nu + \frac{1}{3}\nu^2 + \frac{1}{9}\nu^3 + \frac{23}{81}\nu^4 - \frac{151}{243}\nu^5 + \frac{1}{9}\nu^6 + \mathcal{O}(\mu^{7/3}) \tag{42}$$

and

$$x_2^* = 1 - \mu^* + \nu + \frac{1}{3}\nu^2 - \frac{1}{9}\nu^3 - \frac{31}{81}\nu^4 - \frac{119}{243}\nu^5 - \frac{1}{9}\nu^6 + \mathcal{O}(\mu^{7/3}), \tag{43}$$

These are precisely the forms described by Eqs. 27 and 22 in Sec. 4.4 of Ref. [4], respectively. If the reader is to compare these forms directly, it should be cautioned that there are some sign differences between the forms given above and those found in Ref. [4]. This occurs because Ref. [4] uses a cartesian coordinate system with the L_1 and L_2 points located along the negative x-axis. Those forms can be converted to positive radial coordinates using $x_1^* = 1 - \mu^* - \xi^{(1)}$ (stated prior to Eq. 25) and $x_2^* = 1 - \mu^* + \xi^{(2)}$ (stated prior to Eq. 17) from Sec. 4.4 of Ref. [4].

B. Comparison of Representations for the L_3 Point

Unlike the solutions for the L_1 and L_2 points, the series solution for the L_3 point proceeds in integer powers of μ and is

$$x_3 = 1 + \frac{17}{12}\mu - \frac{1127}{12^4}\mu^3 + \frac{19\,159}{12^5}\mu^4 - \frac{3\,217\,389}{12^7}\mu^5 + \frac{145\,523\,287}{12^8}\mu^6 + \mathcal{O}\left(\mu^7\right). \tag{44}$$

A representation in terms of μ^* can be calculated from the above by using the conversion from μ to μ^* described in the introduction to this section, that is, $\mu = \mu^*/(1-\mu^*) = \sum_{k=1}^{\infty} \mu^{*k}$. In contrast to the forms for the L_1 and L_2 points that presented an expansion in terms of μ, since $\mu = \mu^*/(1-\mu^*)$, the form for the L_3 point provided in Ref. [4] is an expansion in terms of μ^*. As such, the conversion from x to x^* proceeds better by using $x^* = x(1-\mu^*)$. Expressing x_3 as $x_3 = 1 + \sum_{n=1}^{\infty} a_n \mu^n$, these conversions yield $x_3^* = 1 - \mu^* + (1-\mu^*)\sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} \mu^{*k}\right)^n$, which generates the following expression for x_3^*

$$x_3^* = 1 + [a_1 - 1]\,\mu + [a_2]\,\mu^{*2} + [a_2 + a_3]\,\mu^{*3} + [a_2 + 2a_3 + a_4]\,\mu^{*4} \\ + [a_2 + 3a_3 + 3a_4 + a_5]\,\mu^{*5} + [a_2 + 4a_3 + 6a_4 + 4a_5 + a_6]\,\mu^{*6} + \mathcal{O}\left(\mu^{*7}\right). \tag{45}$$

The reader may notice that coefficient a_1 appears only once in the conversion since $(1-\mu^*)\sum_{k=1}^{\infty}\mu^{*k} = \mu^*$, and the expressions for the other coefficients involve the binomial series. Applying these formulas to the coefficients in Eq. 46, and recalling that $a_2 = 0$, yields

$$x_3^* = 1 + \frac{5}{12}\mu^* - \frac{1127}{12^4}\mu^{*3} - \frac{7889}{12^5}\mu^{*4} - \frac{783\,069}{12^7}\mu^{*5} - \frac{7\,594\,363}{12^8}\mu^{*6} + \mathcal{O}(\mu^{*7}). \tag{46}$$

The formula for the location of the L_3 point described in Ref. [4] is given as a series of definitions: $x_3^* = \mu^* + \xi$ with $\xi = 1 + \eta$, which gives $x_3^* = 1 + \mu^* + \eta$. The function η is given in Eq. 34 of sec. 4.4 of Ref. [4] in terms of $\nu = 7\mu^*/12$. Putting all of this together yields the same form as Eq. 46.

C. Comparison of Representations for the L_4 and L_5 Points

Since an exact, closed-form solution exists for the L_4 and L_5 points when described in terms of μ, a similarly simple form will exist when the solution is described in terms of μ^*. Recalling that the solution in terms of μ is

$$x_{4,5} = \sqrt{1 + \mu + \mu^2}, \tag{47}$$

a representation in terms of μ^* can be calculated by using the conversion from μ to μ^* described in the introduction to this section, that is, $\mu = \mu^*/(1-\mu^*)$. Since the end goal here is a description in terms of μ^*, the conversion formula $x^* = x/(1+\mu)$ is better written as $x^* = x(1-\mu^*)$. Applying these conversions yields

$$x_{4,5}^* = \sqrt{1 - \mu^* + \mu^{*2}}. \tag{48}$$

IV. COMPARISON OF VISUAL REPRESENTATIONS OF EQS. [6] AND [7]

The main text describes how the illustrations of the effective potential in other published sources often rely on an alternate form of the effective potential whose centrifugal term is negative. The following figures illustrate this point by comparing the contour plots generated from Eq. [6] with those generated from Eq. [7] of the main text. Logarithmic contour scales are used in both plots and the locations of the Lagrange points are marked with the symbol '×'.

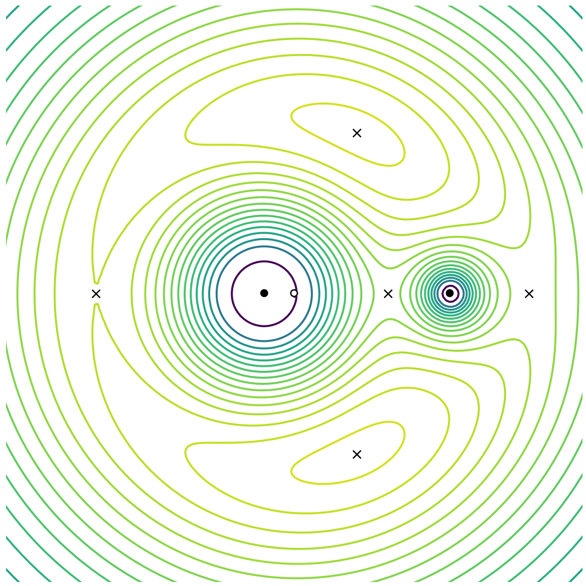

FIG. 9. This contour plot is calculated using Eq. [6] of the main text, which clearly shows the locations of the five Lagrange points.

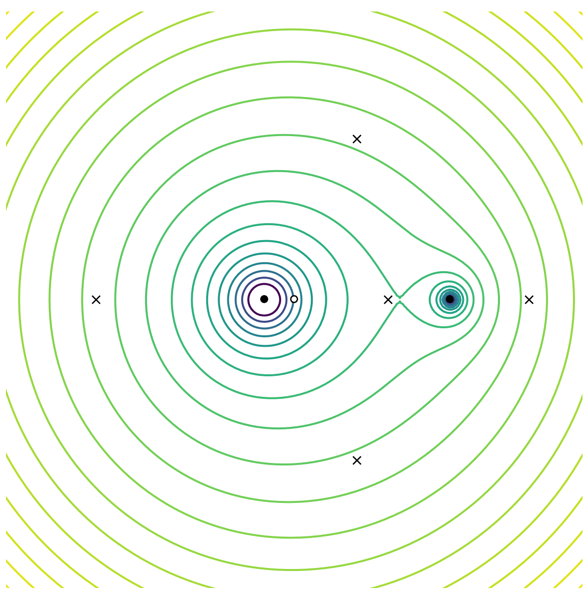

FIG. 10. This contour plot is calculated using Eq. [7] of the main text with the substitution $L = m_3 r^2 \omega_0$, showing that no information about the five equilibrium points of the system is revealed.