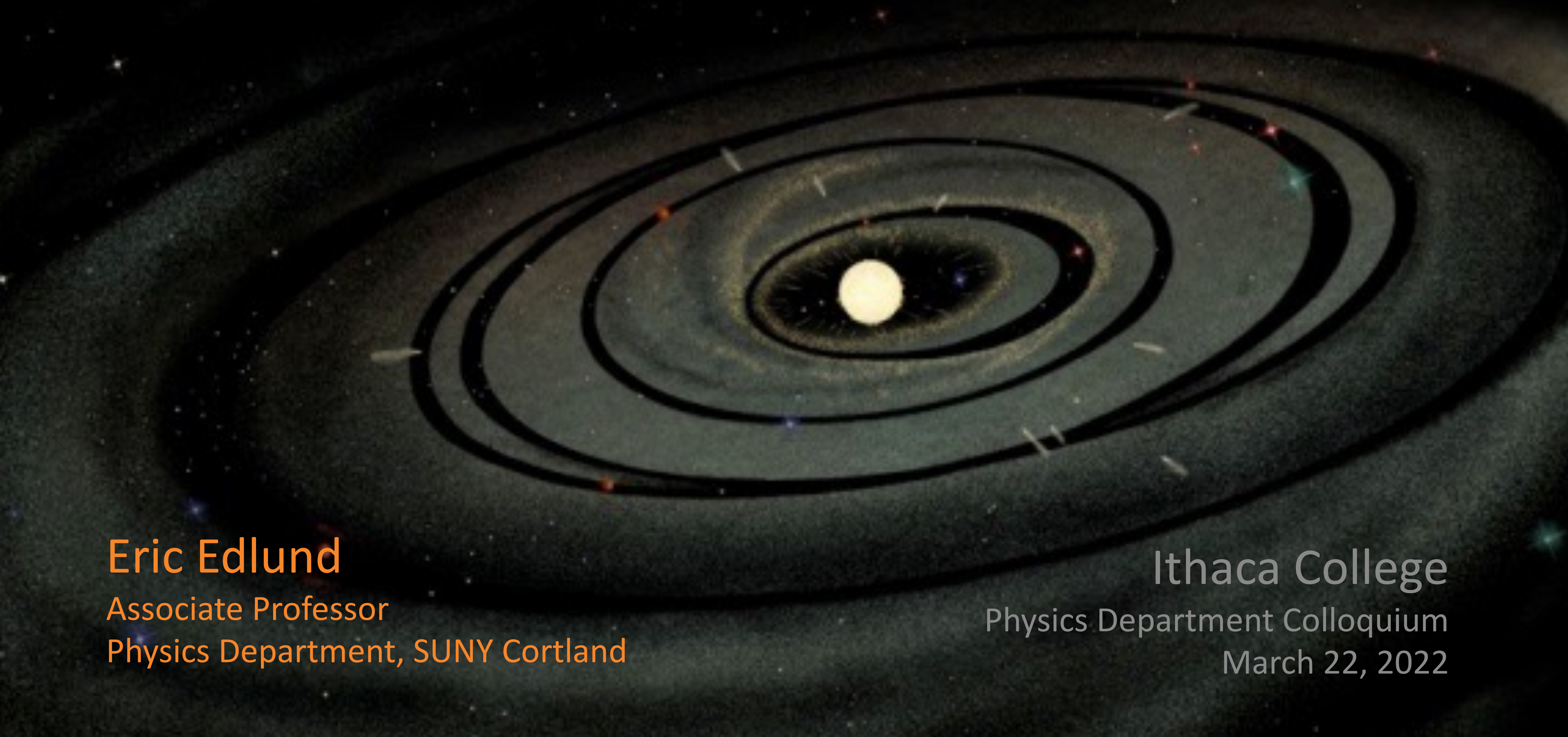
# Calculations for orbital intercept & rendezvous



Eric Edlund
Associate Professor
Physics Department, SUNY Cortland

Ithaca College
Physics Department Colloquium
March 22, 2022

# Real life interception scenarios

image credit: cortlandvoice.com


image credit: spacenews.com

image credit: foreignpolicy.com

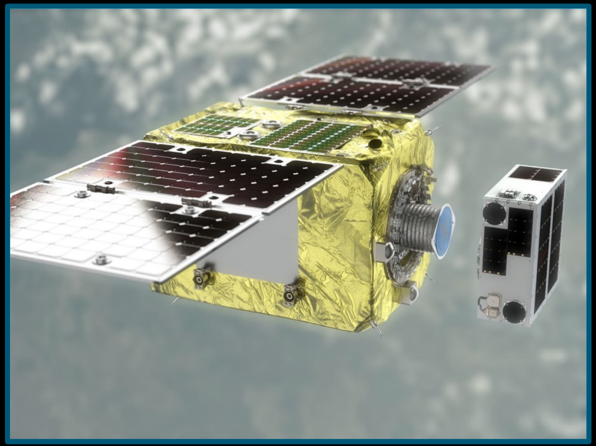
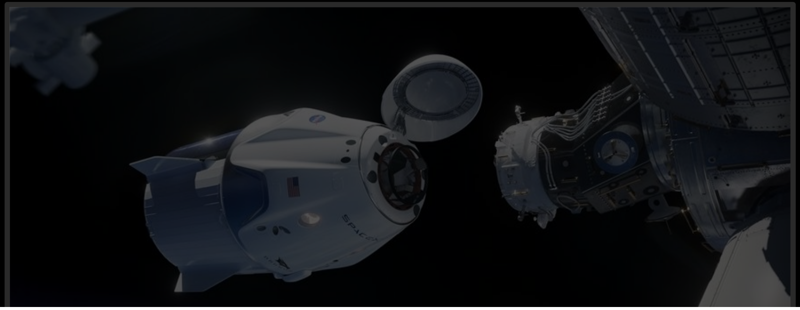


image credit: www.smithsonianmag.com

# Real life interception scenarios

image credit: cortlandvoice.com

image credit: spacenews.com



## Maneuvering Russian Satellite Has Everyone's Attention

by **Mike Gruss** — July 17, 2015

spacenews.com

image credit: foreignpolicy.com

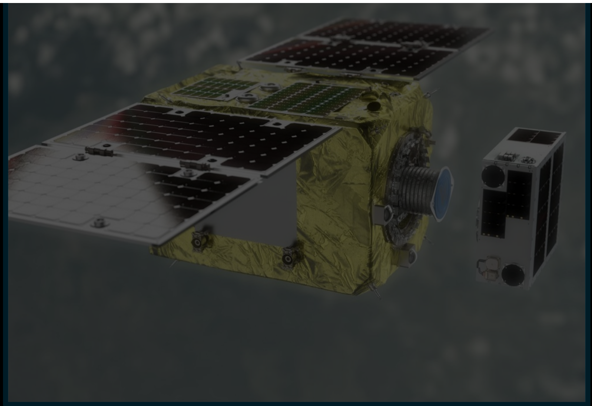


image credit: www.smithsonianmag.com

# The plan for this talk

- Introduction & background
  - A few escape velocity problems
  - The Lambert problem
  - Our intercept problem

- Formulating a general solution
  - Formulation of the problem
  - Equations of motion -> intercept conditions
  - Relationship to control parameters

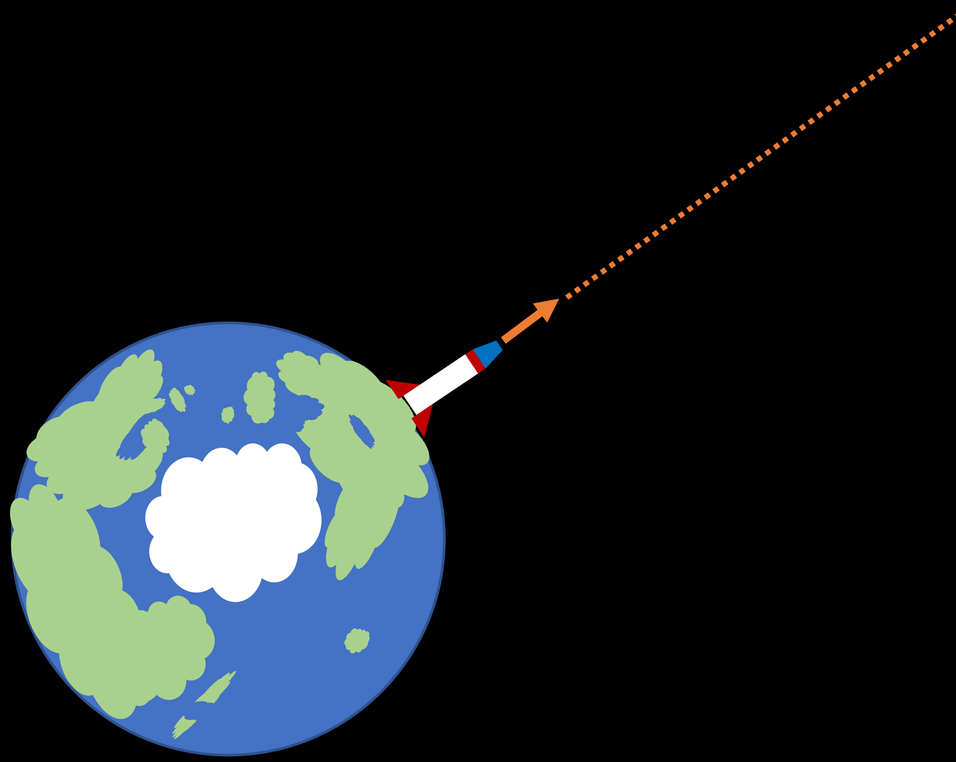
- Specific solutions
  - Intercept at the origin
  - Extension to fast-intercepts

# An introductory problem: escape velocity

- What thrust is needed for a craft to escape from Earth?

$$\frac{1}{2}m(\Delta v)^2 - \frac{GMm}{r_0} = \frac{1}{2}\cancelto{0}{mv_f^2} - \cancelto{0}{\frac{GMm}{r_f}}$$

- Let’s consider three cases:
  1. direct launch from Earth
  2. Parallel thrust from orbit
  3. Perpendicular thrust from orbit



# An introductory problem: escape velocity

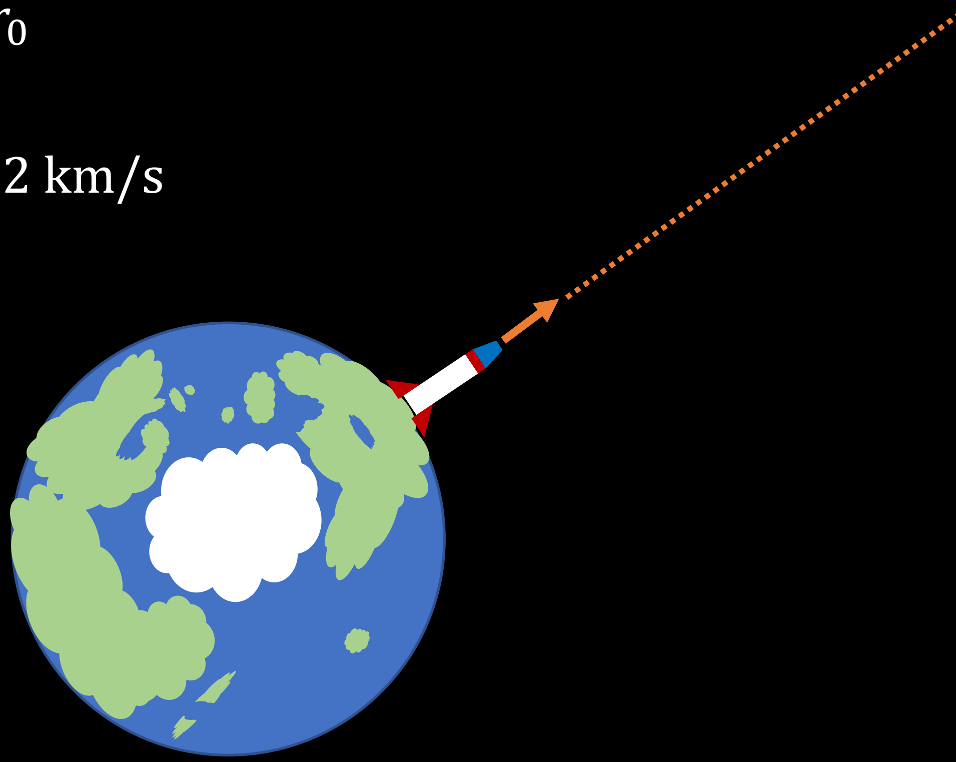
- What thrust is needed for a craft to escape from Earth?

- Let's consider three cases:
    1. direct launch from Earth (11.2)
    2. Parallel thrust from orbit
    3. Perpendicular thrust from orbit

$$\frac{1}{2}m(\Delta v)^2 - \frac{GMm}{r_0} = 0$$

$$\Delta v = \sqrt{\frac{2GM}{r_0}}$$

$$\Delta v_E \approx 11.2 \text{ km/s}$$



# An introductory problem: escape velocity

- What thrust is needed for a craft to escape from Earth?

- Let's consider three cases:
  1. direct launch from Earth (11.2)
  2. Parallel thrust from orbit (3.3)
  3. Perpendicular thrust from orbit

$$\frac{1}{2}m(v_0+\Delta v)^2-\frac{GMm}{r_0}=0$$

$$\Delta v=\sqrt{\frac{2GM}{r_0}}-\sqrt{\frac{GM}{r_0}}=\sqrt{\frac{2GM}{r_0}}\left(1-\frac{1}{\sqrt{2}}\right)$$

$$\Delta v_E\approx 3.3\ \mathrm{km/s}$$

$$v_0^2=\frac{GM}{r_0}$$

# An introductory problem: escape velocity

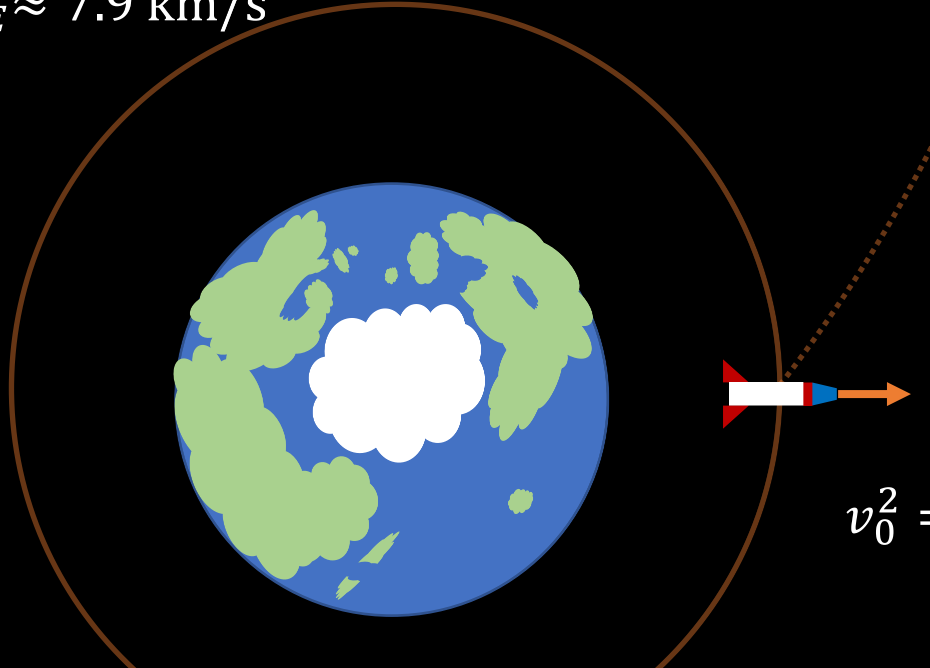
- What thrust is needed for a craft to escape from Earth?

$$\frac{1}{2}m(v_0^2 + \Delta v^2) - \frac{GMm}{r_0} = 0$$

$$\Delta v = \sqrt{\frac{2GM}{r_0} - \frac{GM}{r_0}} = \sqrt{\frac{2GM}{r_0}}\frac{1}{\sqrt{2}}$$

- Let’s consider three cases:
  1. direct launch from Earth (11.2)
  2. Parallel thrust from orbit (3.3)
  3. Perpendicular thrust from orbit (7.9)

$$\Delta v_E \approx 7.9 \text{ km/s}$$



$$v_0^2 = \frac{GM}{r_0}$$

- Next up: a challenge question

# An orbital dynamics conundrum

- All paths to r = ∞ require the same work by the engines. Equivalently, the total energy is a conserved quantity precisely because the work is path independent.

$$\Delta v = v_{exh} \ln\left(\frac{m_i}{m_f}\right)$$

- The rocket equation tells us how $\Delta v$ is related to a change in mass of the rocket

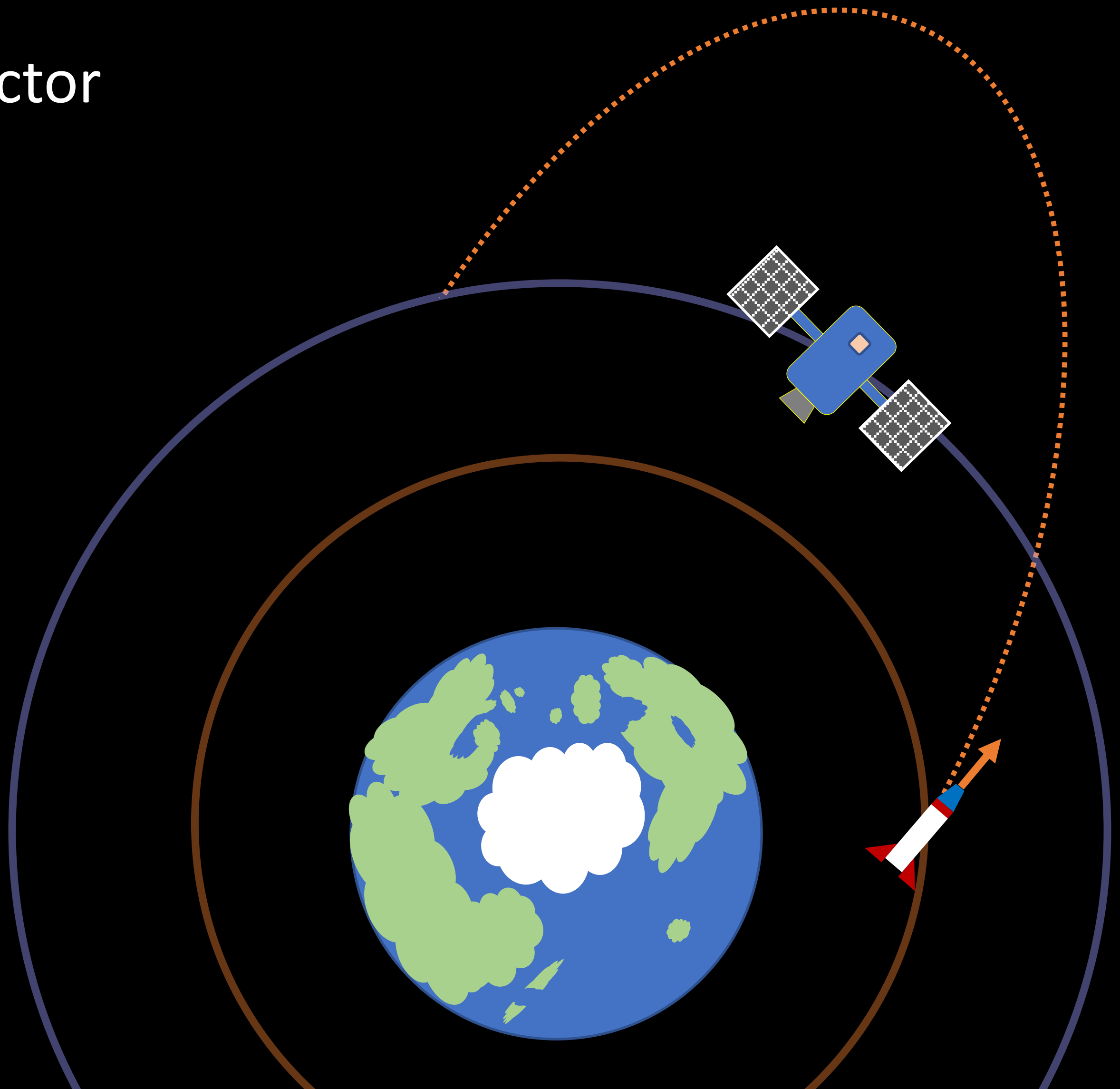
$$m_f = m_i e^{-\Delta v / v_{exh}}$$

- The prior analysis indicates that a different $\Delta v$, and therefore a different quantity of fuel and chemical energy, is required for each path.

- It cannot be simultaneously true that the work is the same for all paths and yet each path requires a different chemical energy.

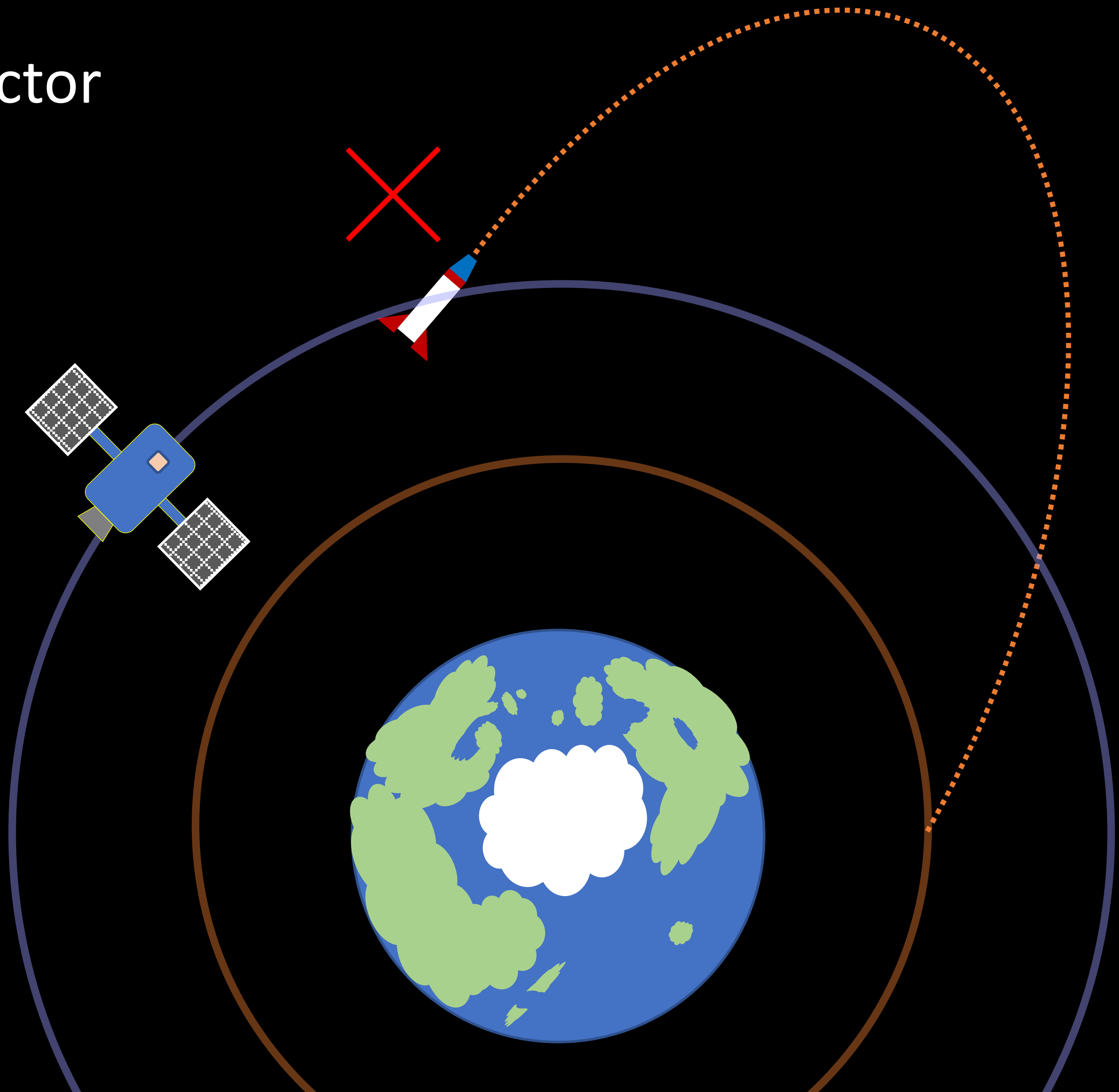
# The main event: Interception and Rendezvous

- The first question: What thrust vector will allow us to intercept a target?



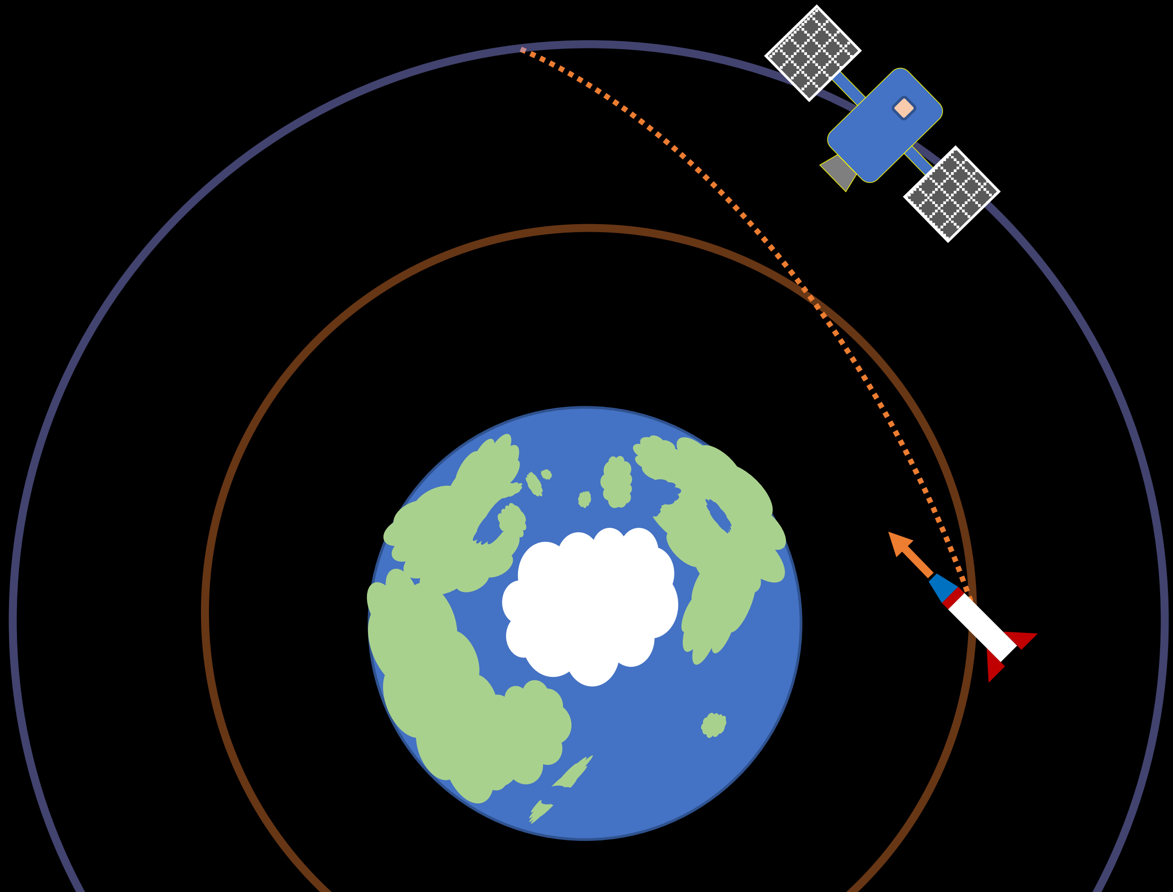
# The main event: Interception and Rendezvous

- The first question: What thrust vector will allow us to intercept a target?



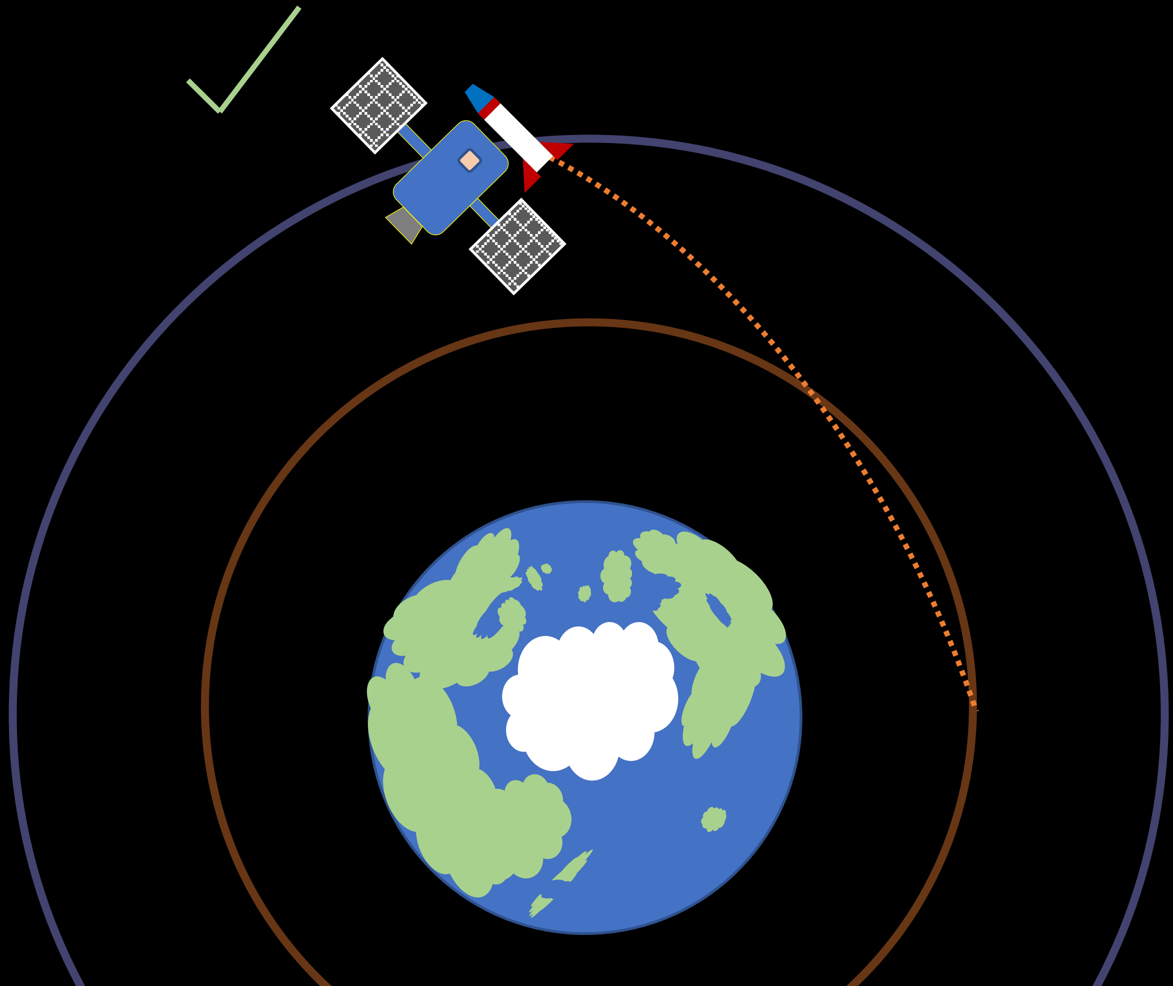
# The main event: Interception and Rendezvous

- The first question: What thrust vector will allow us to intercept a target?



# The main event: Interception and Rendezvous

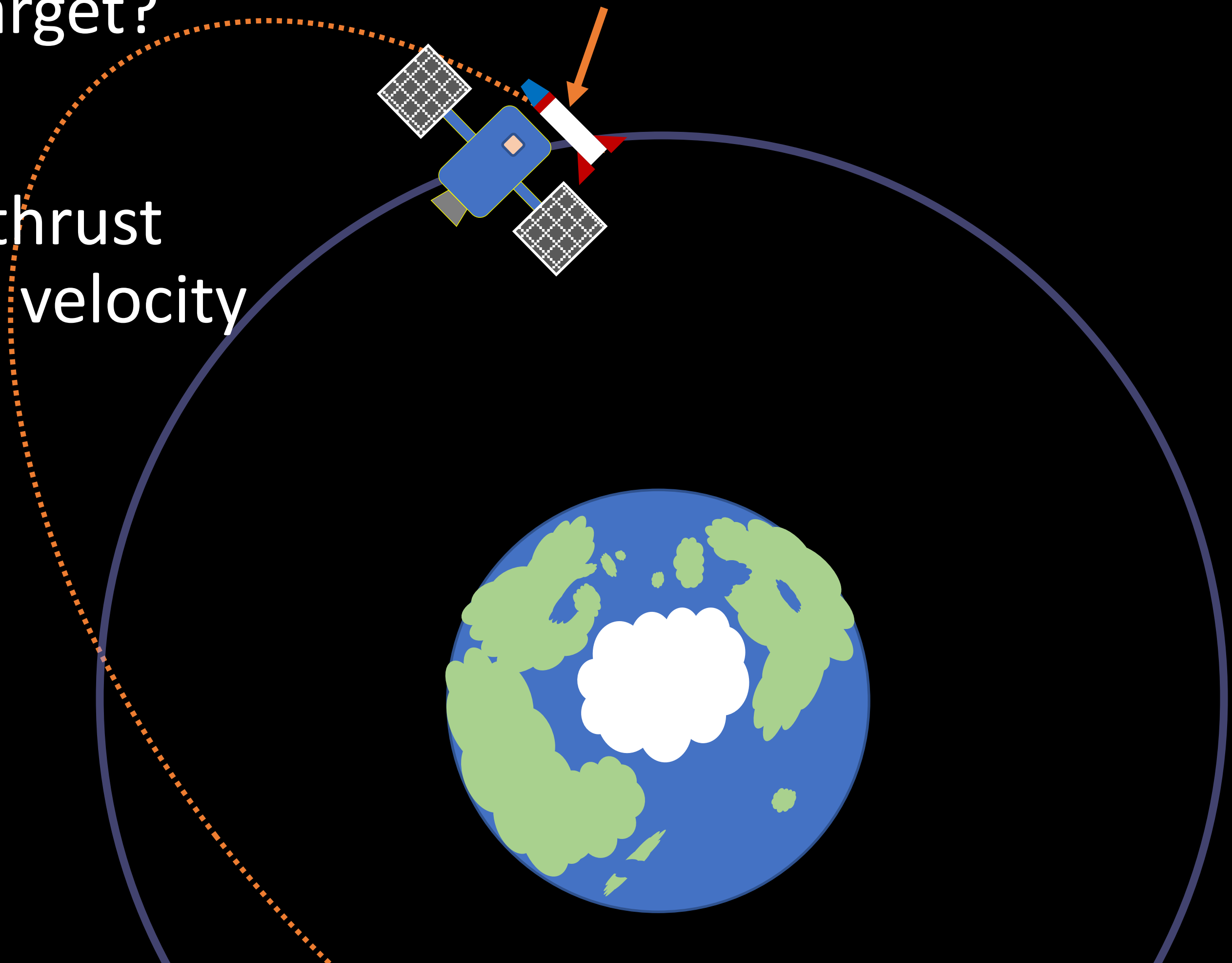
- The first question: What thrust vector will allow us to intercept a target?



# The main event: Interception and Rendezvous

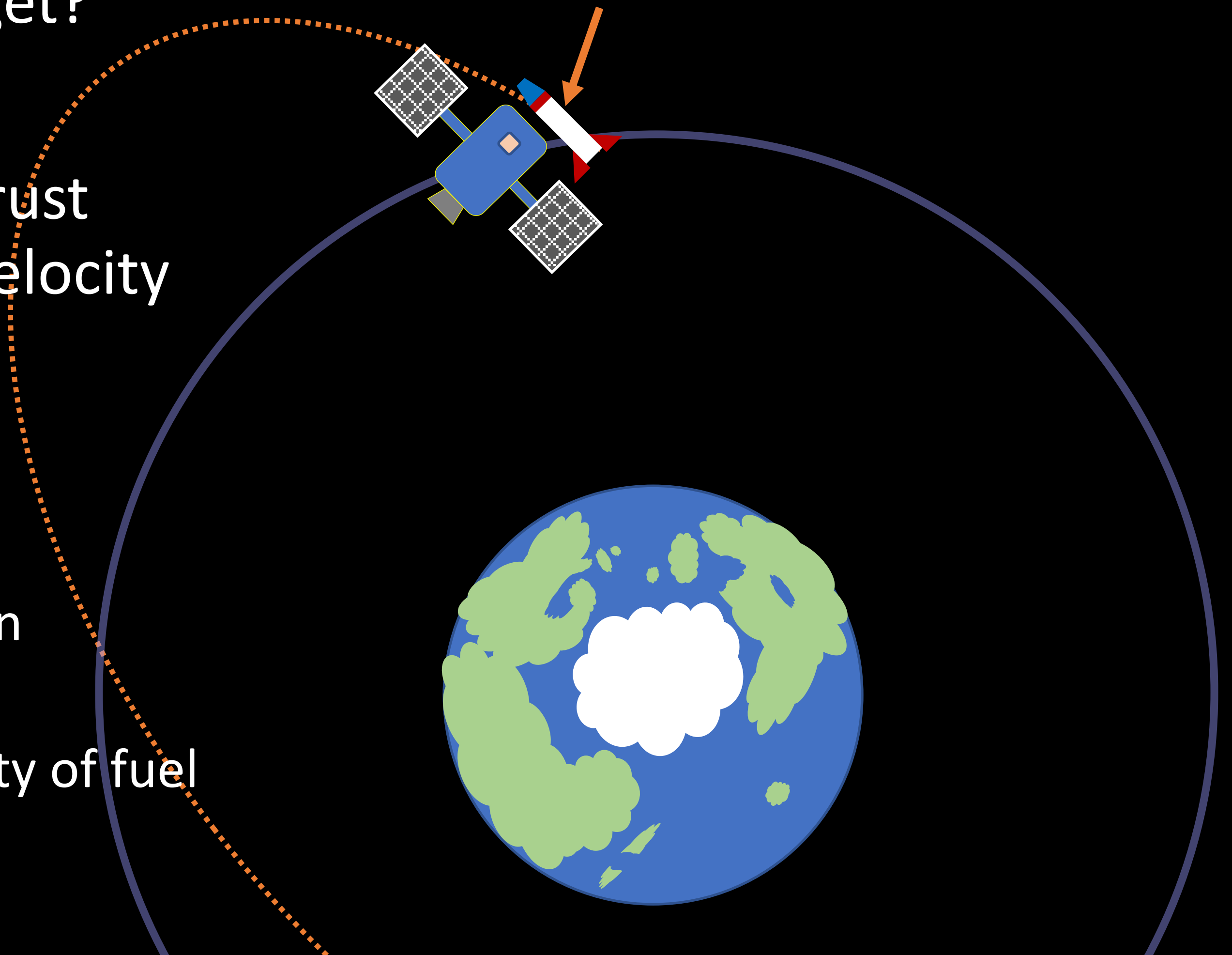
- The first question: What thrust vector will allow us to intercept a target?

- The second question: What thrust vector will allow us to match velocity with the target?



# The main event: Interception and Rendezvous

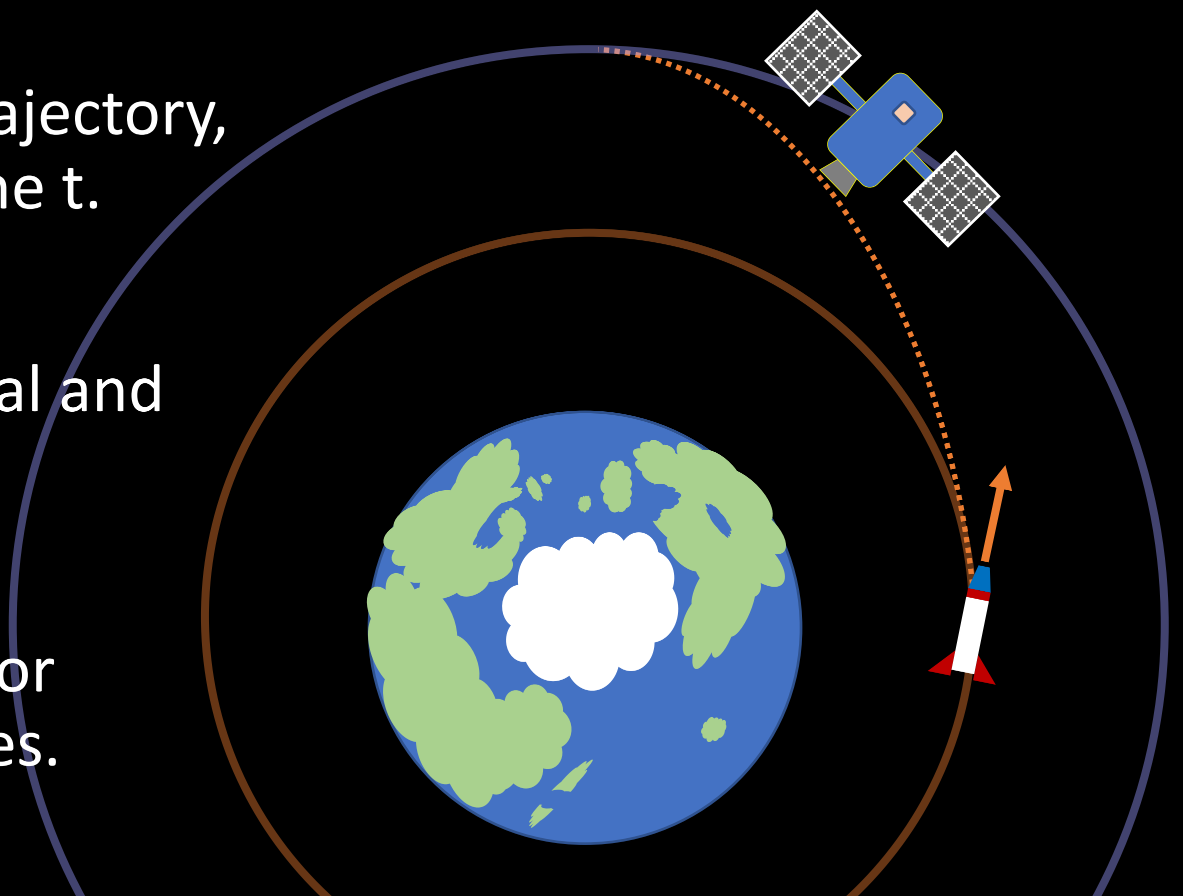
- The first question: What thrust vector will allow us to intercept a target?
- The second question: What thrust vector will allow us to match velocity with the target?
- What are your constraints?
  - Interception at a specific position
  - Interception at a specific time
  - Interception with a given quantity of fuel
  - . . .



# The Lambert problem

| | |
|---|---|
| Geometric solution (Lambert): | 1761 |
| Formal proof (Lagrange): | 1788 |
| Numerical solutions (Gauss): | 1857 |
| Robust algorithms: | 1950’s |
| Modern improvements: | ongoing |

- What thrust vector will allow us to intercept the target at a given time?
- The target moves on an inertial trajectory, so we know where it will be at time t.
- We have two constraints: the radial and angular coordinates at time t.
- We are looking for the thrust vector (magnitude & direction) that solves.



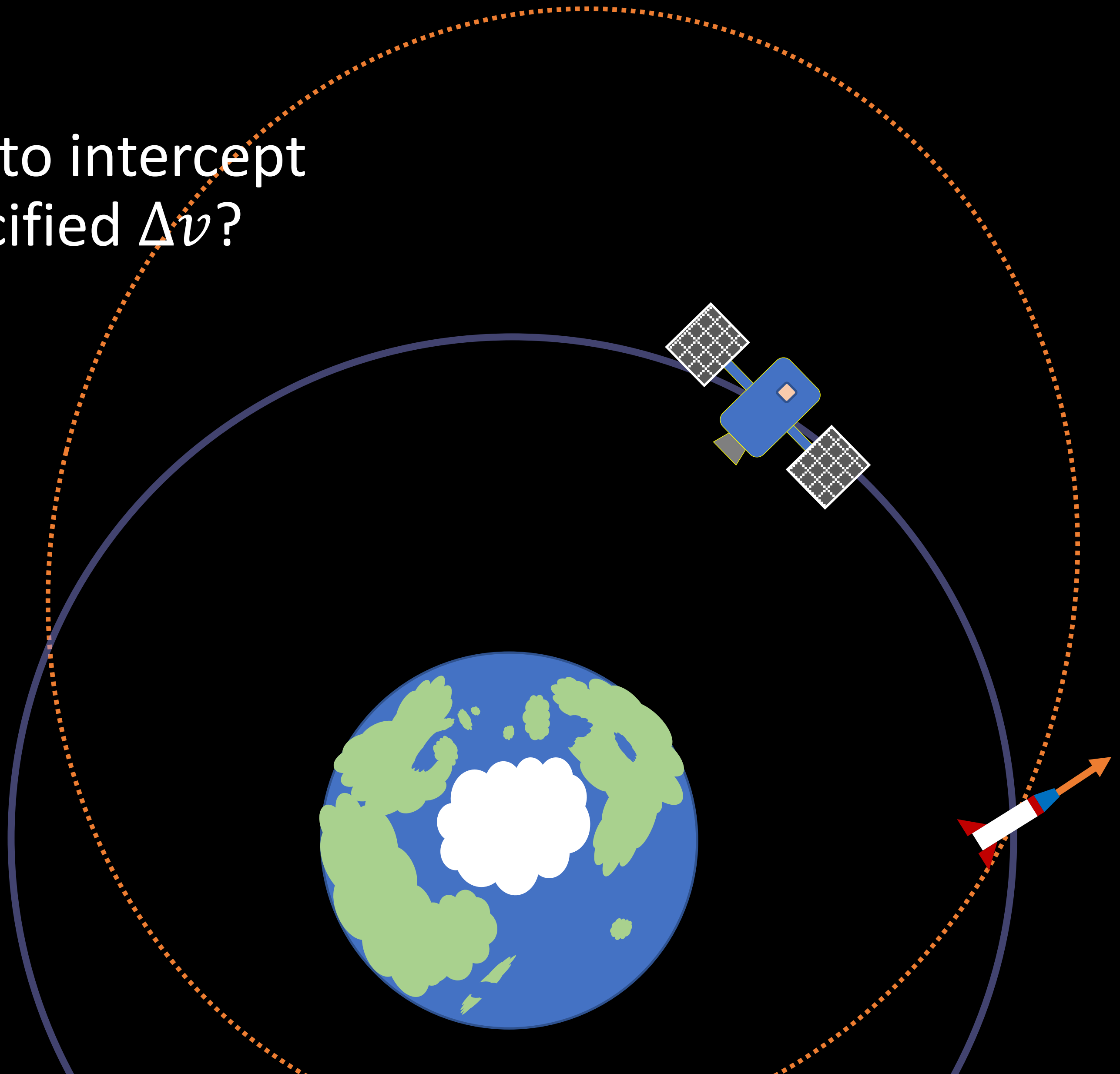
# The Edlund problem

- What thrust angle will allow us to intercept the target when we have a specified $\Delta v$?

- The target is inertial, so its equation of motion is known.

$$\theta_{target}(t) = \text{known}$$
$$r_{target}(t) = \text{known}$$

- The chaser can actively modify its trajectory. We need $\theta_{chaser}(t)$ & $r_{chaser}(t)$



# Part 1: formulating the problem

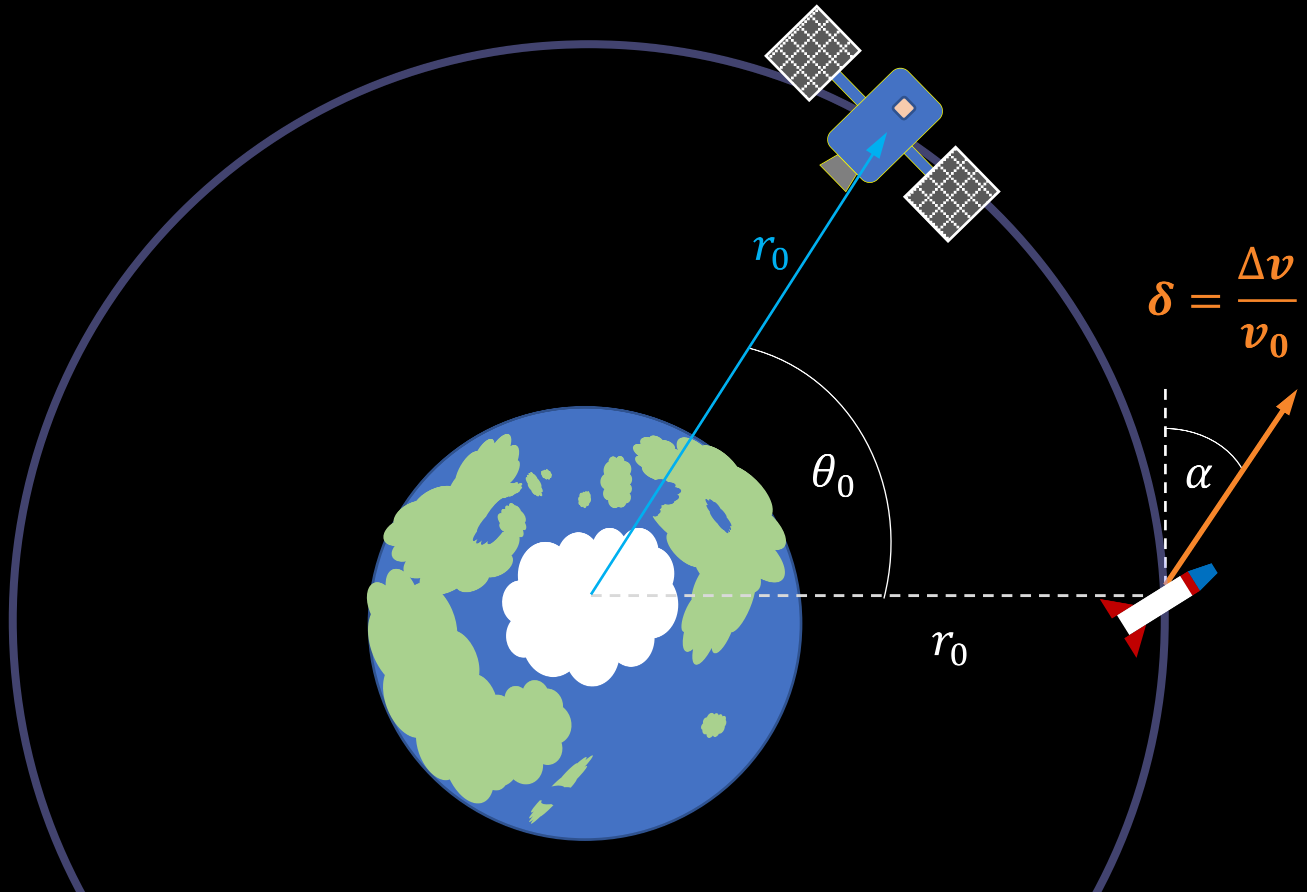
- Intercept means that both space craft have the same coordinates.

- That is, we require:

$$r_{\text{chaser}} = r_{\text{target}}$$

$$\theta_{\text{chaser}} = \theta_{\text{target}}$$

(modulo $2\pi$)

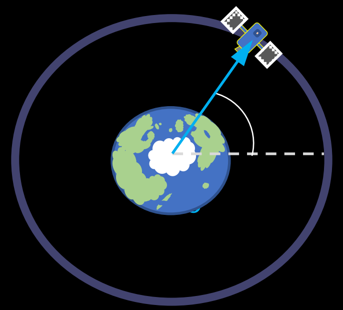


# Part 2: the equations of motion

- The target:

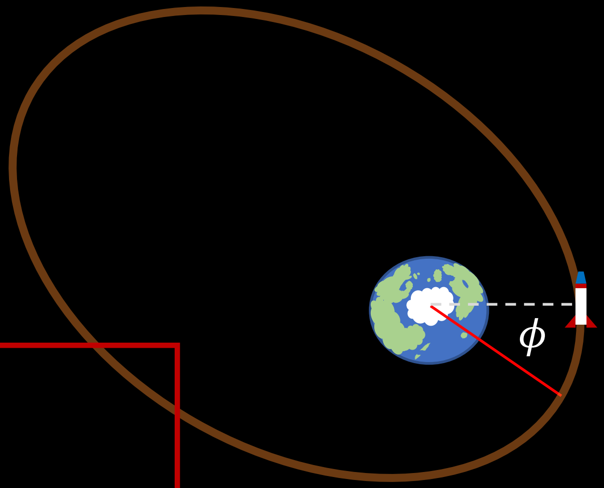
$$r_t(t) = r_0 \;\; (\text{constant})$$

$$\theta_t(t) = \theta_0 + \omega_0 t$$



- The chaser:

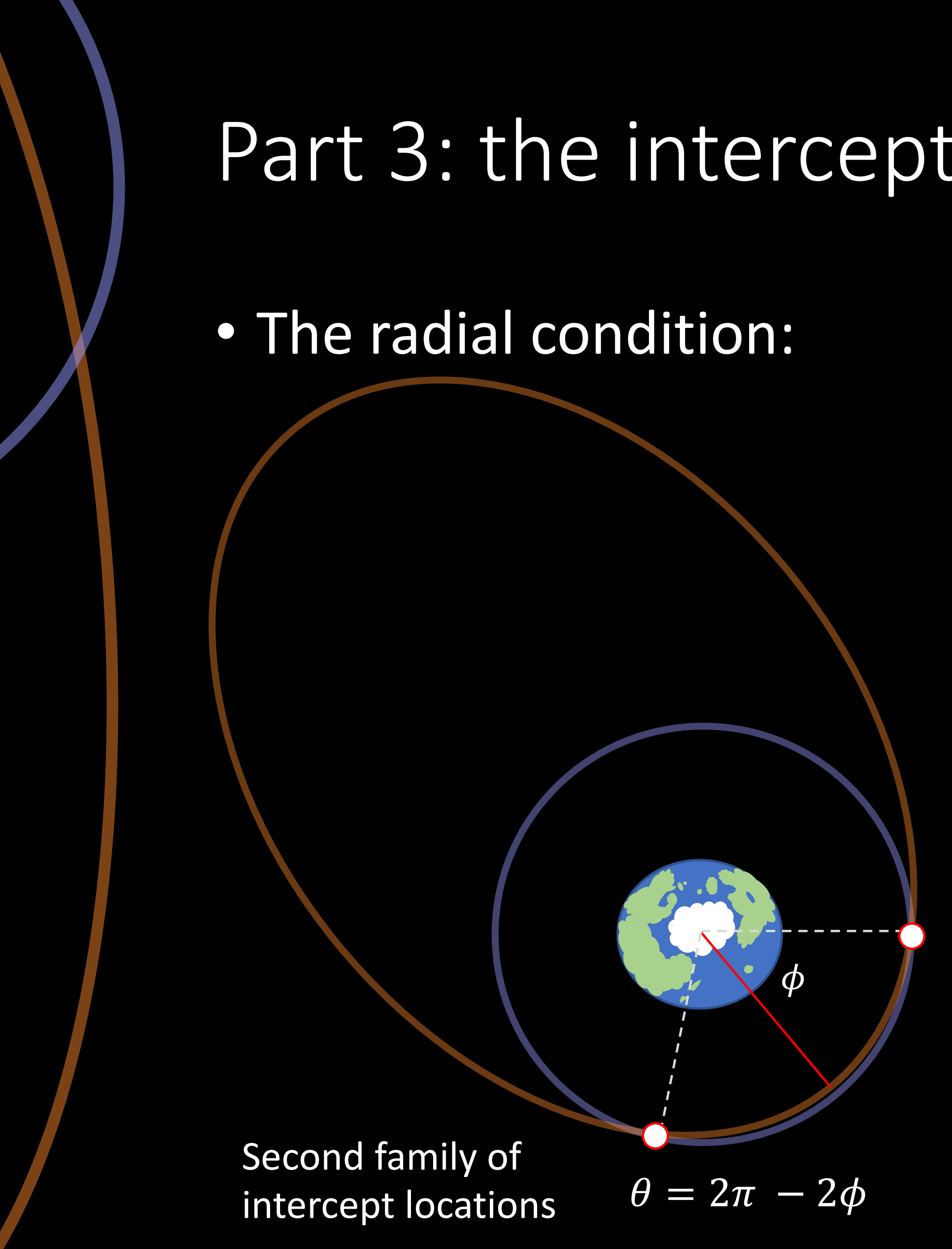
$$r_c(\theta_c) = r_0 \frac{1 + \epsilon\cos(\phi)}{1 + \epsilon\cos(\theta_c + \phi)}$$




$$[1 + \epsilon\cos(\phi)]^2 \int_0^{\theta_c} \frac{\mathrm{d}\theta}{[1 + \epsilon\cos(\theta + \phi)]^2} = \omega_0 t$$

# Part 3: the intercept condition

- The radial condition:



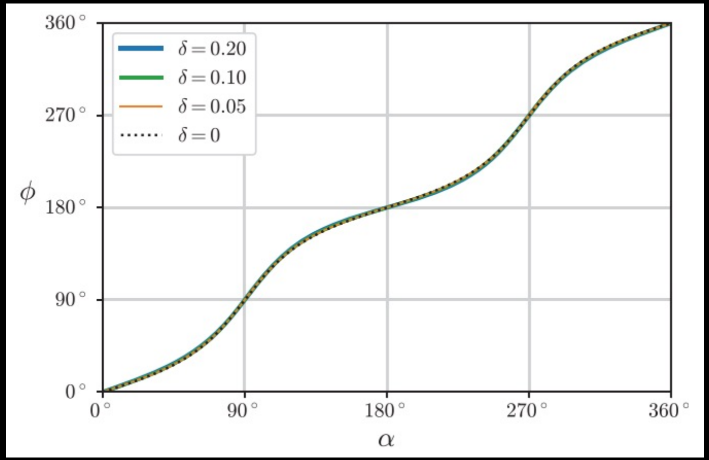

- The angular condition:

$$\theta_t(\theta_c) \stackrel{!}{=} \theta_0 + [1 + \epsilon\cos(\phi)]^2 \int_0^{\theta_c} \frac{\mathrm{d}\,\theta}{[1 + \epsilon\cos(\theta + \phi)]^2}$$

$$2\pi n_t \stackrel{!}{=} \theta_0 + 2\pi n_c \left[\frac{1 + \epsilon\cos(\phi)}{1 - \epsilon^2}\right]^{3/2}$$

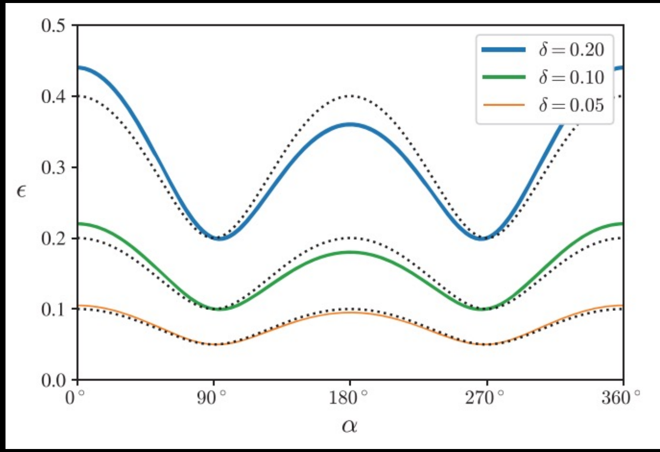
# Part 4: relating the ellipse & control parameters

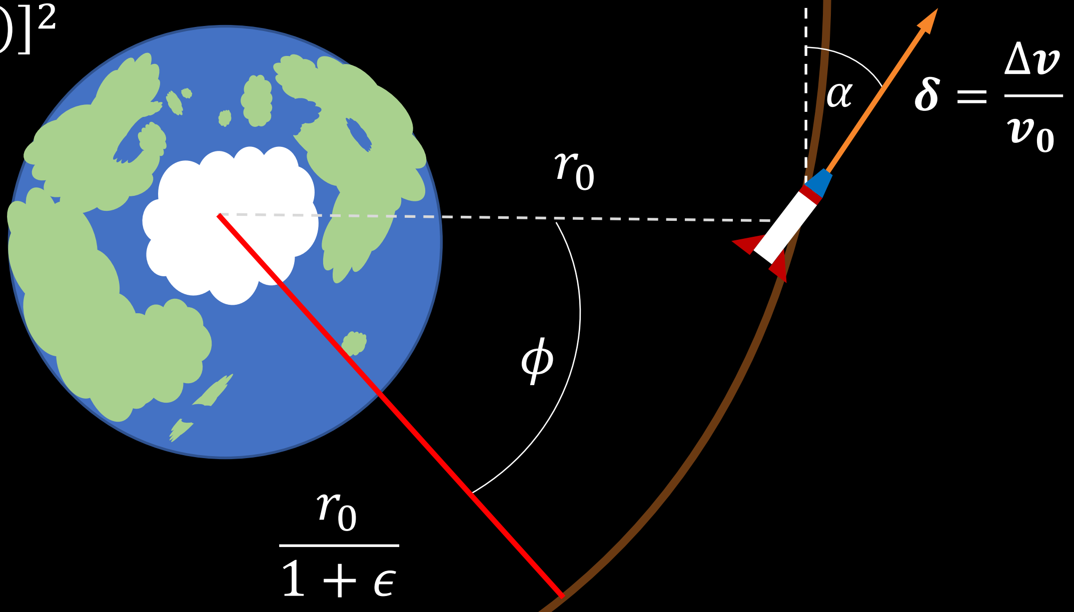
$$\tan(\phi) = \frac{1 + \delta\cos(\alpha)}{2 + \delta\cos(\alpha)}\tan(\alpha)$$




$$\tan(\phi) \approx \frac{1}{2}\tan(\alpha)$$

$$\epsilon = \delta\sqrt{\sin^2(\alpha)[1 + \delta\cos(\alpha)]^2 + \cos^2(\alpha)[2 + \delta\cos(\alpha)]^2}$$




$$\epsilon \approx \delta\sqrt{\sin^2(\alpha) + 4\cos^2(\alpha)}$$

# Part 5: putting it all together

- The intercept condition:

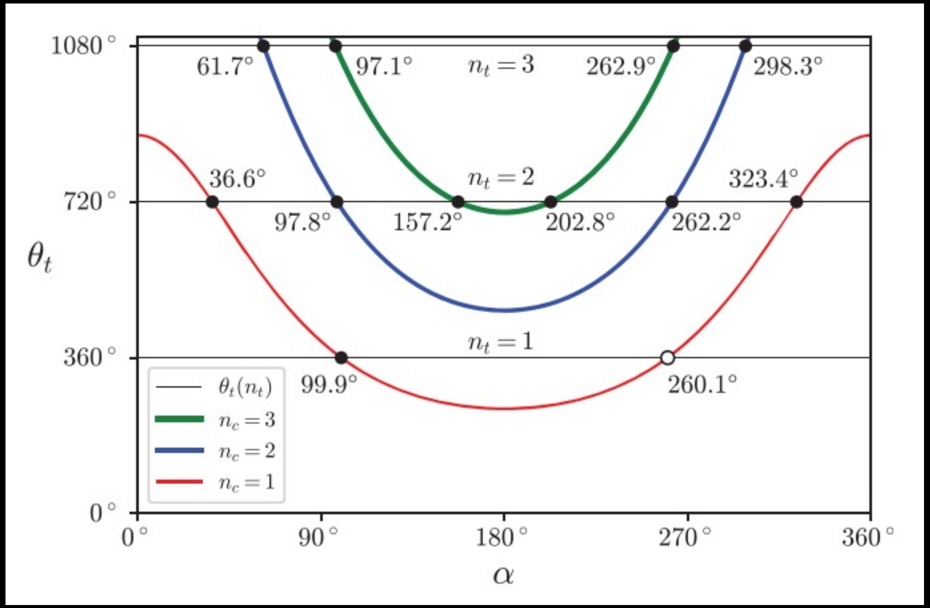
$$2\pi n_t \stackrel{!}{=} \theta_0 + 2\pi n_c \left[\frac{1 + \epsilon\cos(\phi)}{1 - \epsilon^2}\right]^{3/2}$$

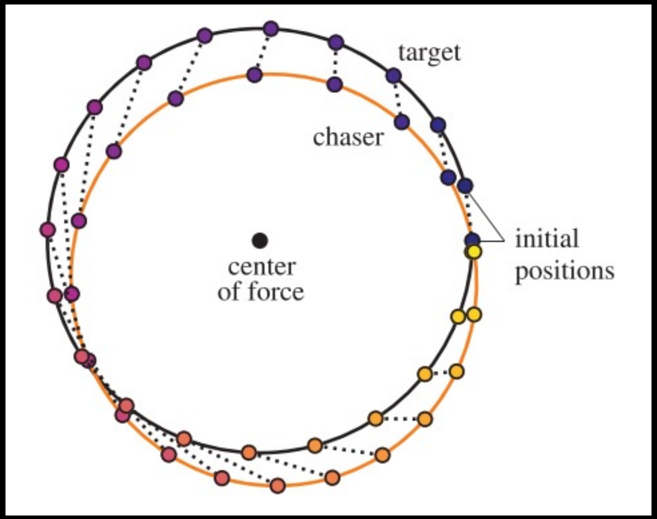
- The control variable relationships:

$$\tan(\phi) = \tan(\alpha)\frac{1 + \delta\cos(\alpha)}{2 + \delta\cos(\alpha)}$$

$$\epsilon = \delta\sqrt{\sin^2(\alpha)[1 + \delta\cos(\alpha)]^2 + \cos^2(\alpha)[2 + \delta\cos(\alpha)]^2}$$

$\theta_0 = 15°$ $\delta = 0.20$

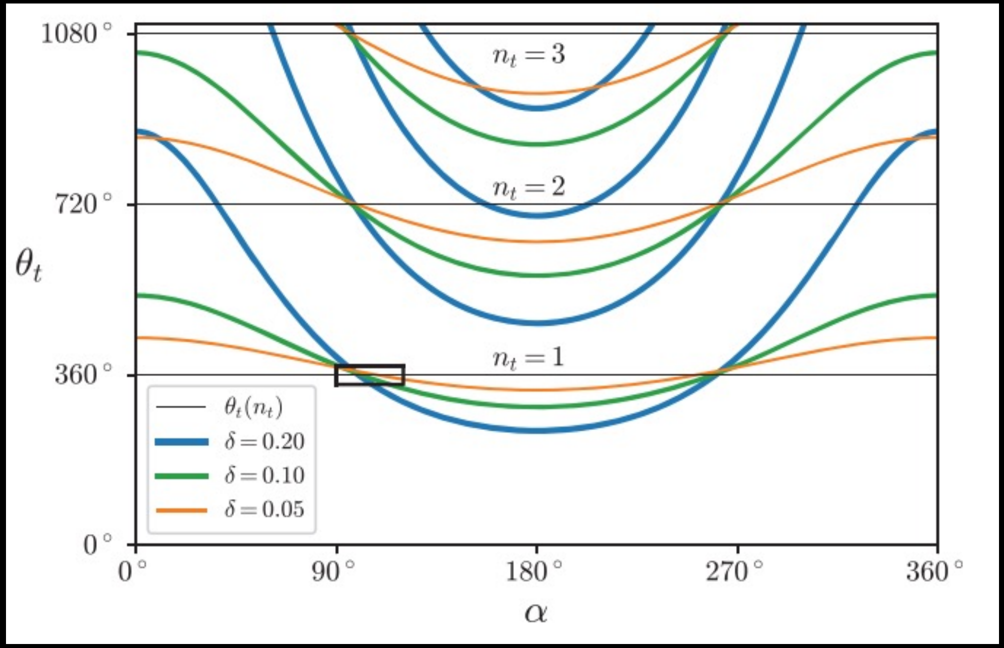


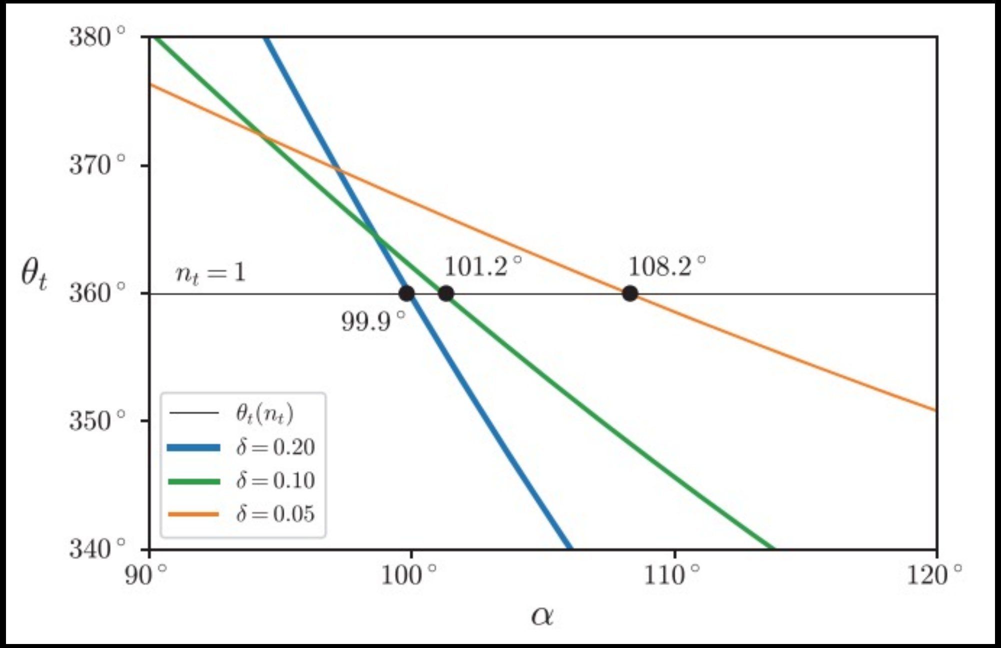



# Part 6: solution stability

- Which solutions are stable with respect to small variations in the thrust angle?

Answer: not solutions near $\alpha = 90°$



# Part 6: solution stability

- Which solutions are stable with respect to small variations in the thrust angle?

Answer: not solutions near $\alpha = 90°$

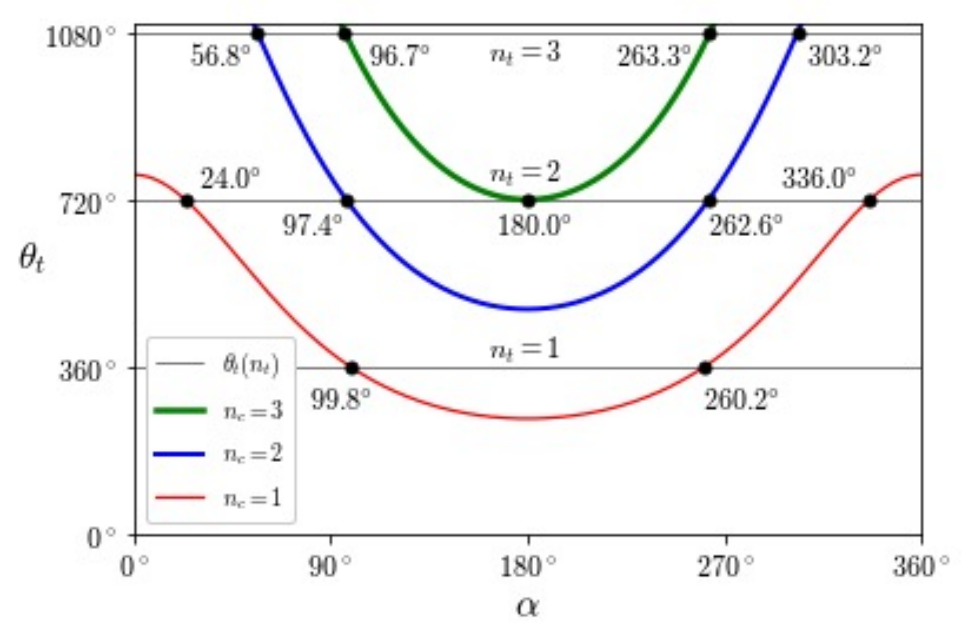
So, what is the answer?

# Part 6: solution stability

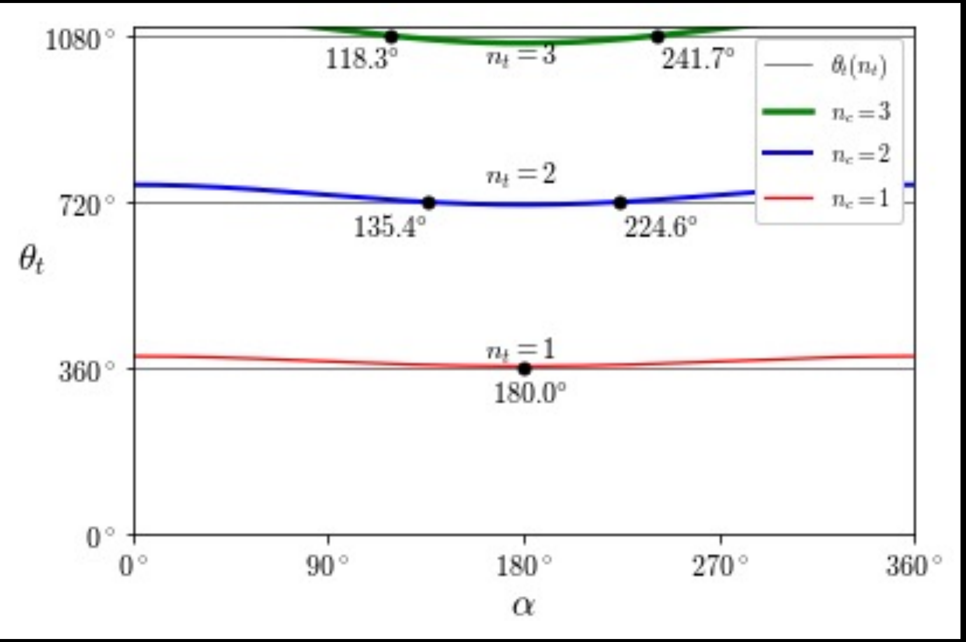
- Which solutions are stable with respect to small variations in the thrust angle?

Answer: not solutions near $\alpha = 90°$

$\theta_0 = 15° \qquad \delta = 0.18$

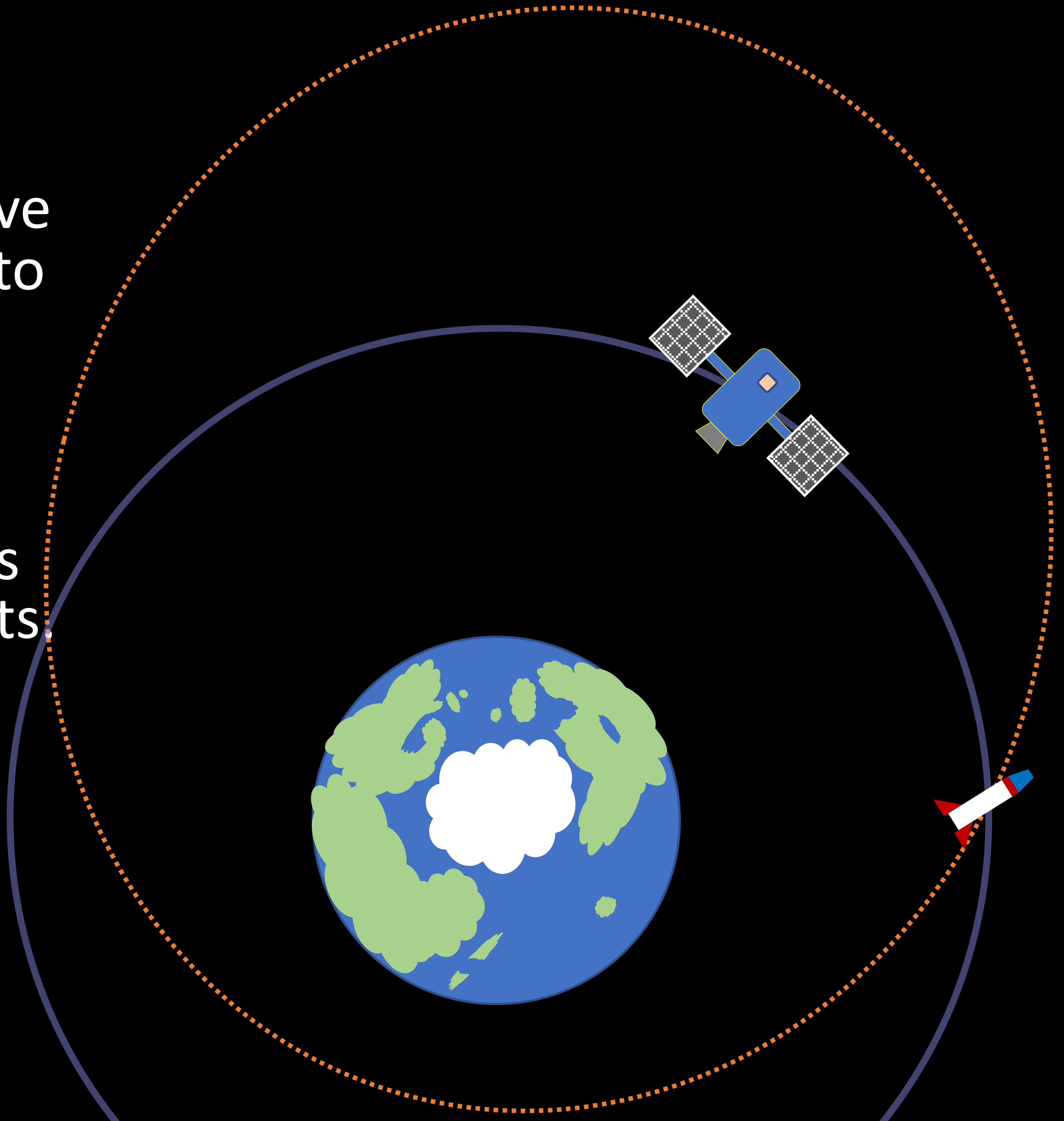



$\theta_0 = 15° \qquad \delta = 0.01$



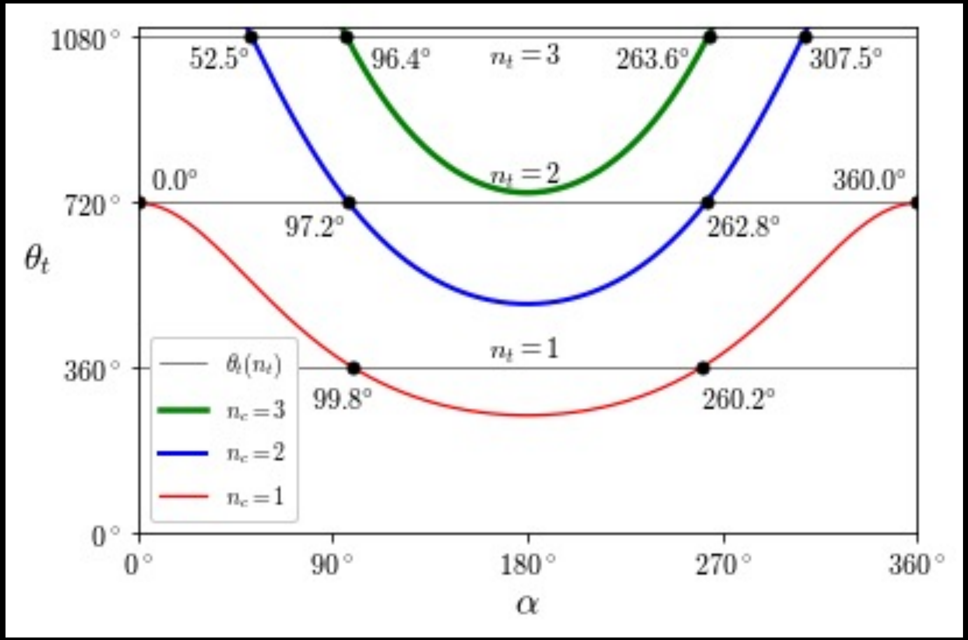
# Conclusions

- The field of orbital dynamics is alive and evolving, with new solutions to old problems.
- The interception & rendezvous problems present great challenges for undergraduate physics students.
- Take care when formulating your questions, constraints, and conclusions!



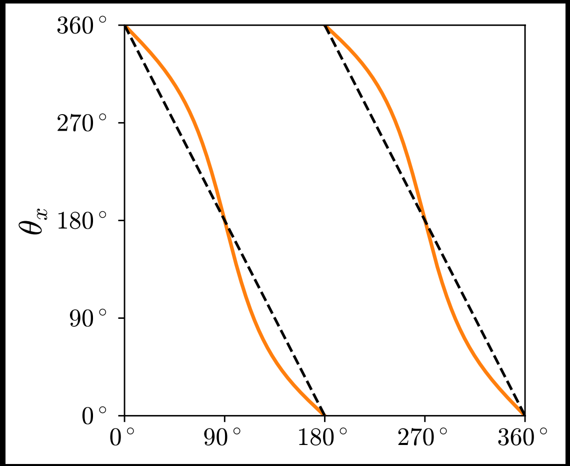
# We can also generate stable solutions for $\alpha = 0°$

$\theta_0 = 15°$ $\delta = 0.166$



# Extension to fast intercept maneuvers

- Second family of intercept locations: $\theta_x = 2\pi - 2\phi$




- The intercept condition:

$$2\pi n_t - 2\phi \overset{!}{=} \theta_0$$

$$+ [1 + \epsilon \cos(\phi)]^2 \int_0^{2\pi n_c - 2\phi} \frac{\mathrm{d}\,\theta}{[1 + \epsilon \cos(\theta + \phi)]^2}$$

$\theta_0 = 15°$ $\delta = 0.20$