Branches, paths, and junctions: what do electrons know of free will?

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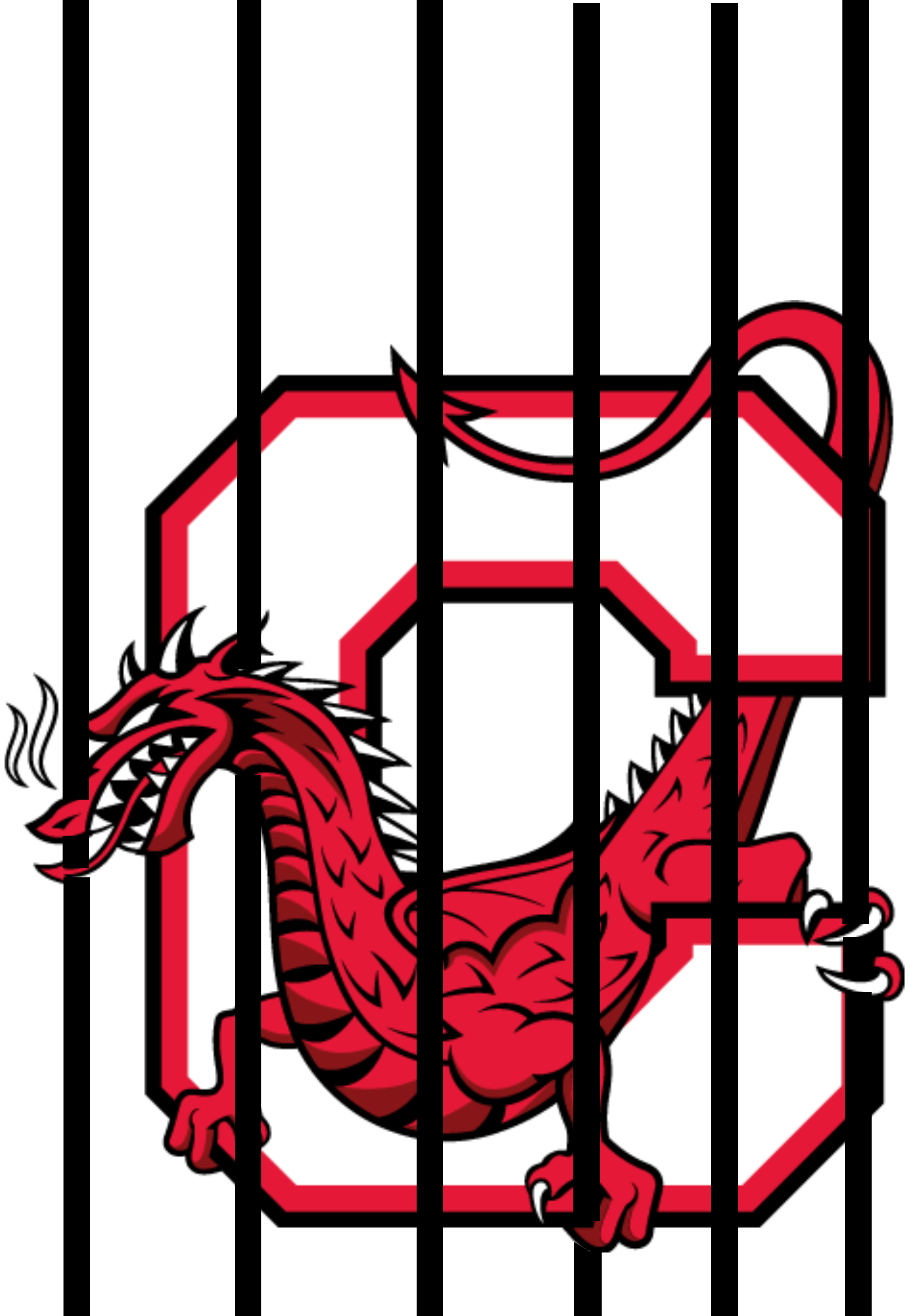
The prisoner’s dilemma

1. Two prisoners are each given a choice:
   - remain loyal and deny the accusations; or
   - defect and incriminate the other prisoner.

2. The penalty for each depends on the choices of both prisoners.

3. A question we might want to answer:
   What is the best action for each prisoner?



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NON-COOPERATIVE GAMES

JOHN NASH

(Received October 11, 1950)

Introduction

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book *Theory of Games and Economic Behavior*. This book also contains a theory of n-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the *absence* of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an *equilibrium point* is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing "good strategies."

Qualities of a Nash equilibrium

- Players act independently and are only concerned with self-interest.
- An equilibrium is defined by a set of choices such that any deviation in a strategy will necessarily result in a worse payoff.
- An equilibrium strategy is often a sub-optimal strategy, that is, equilibrium does not mean that it offers the best individual outcome or the best overall outcome for the group.

The prisoner’s dilemma – equilibrium analysis

- The question that Nash addressed was:
  - Assuming both players play their best strategies, what decision can only yield a worse payoff if a deviation is made?

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Prisoner B choices

Prisoner A choices

	loyal	defect
loyal	**2**	**10**
defect	**0**	**4**

WLOG, since the game is the same for both, let us analyze from the perspective of Prisoner A.

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Nash equilibrium point

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If Prisoner B chooses to remain loyal, then Prisoner A's best strategy is to defect.

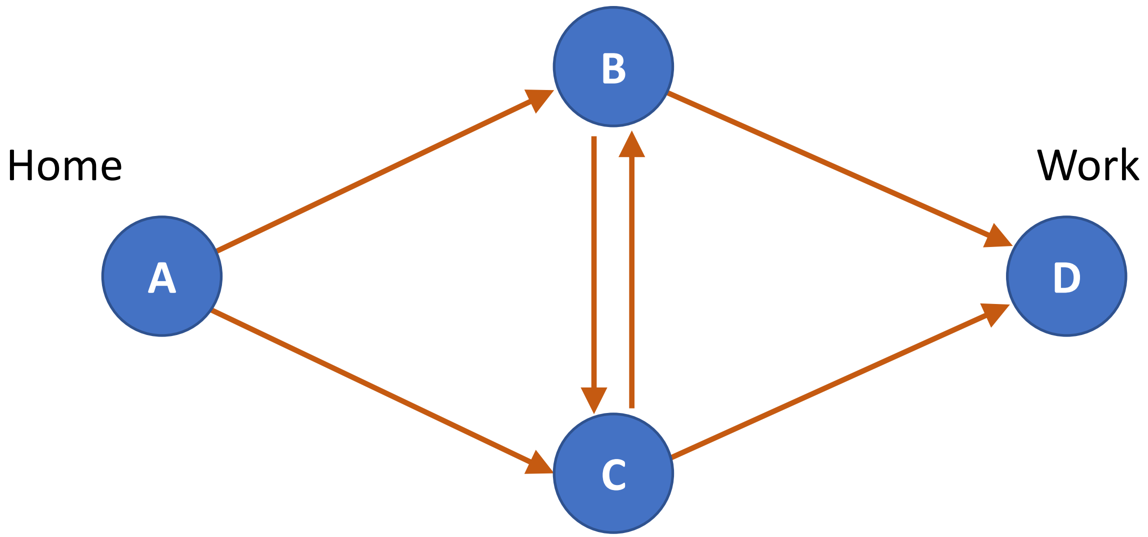
If Prisoner B chooses to defect, then Prisoner A's best strategy is to also defect.

Therefore, both prisoners should always defect.

N-person games of independent commuters

The conditions of this game:

- You can choose from multiple paths to commute from home to work.
- The commute time for each path has a function associated with it.
- Each commuter acts independently and without communication.



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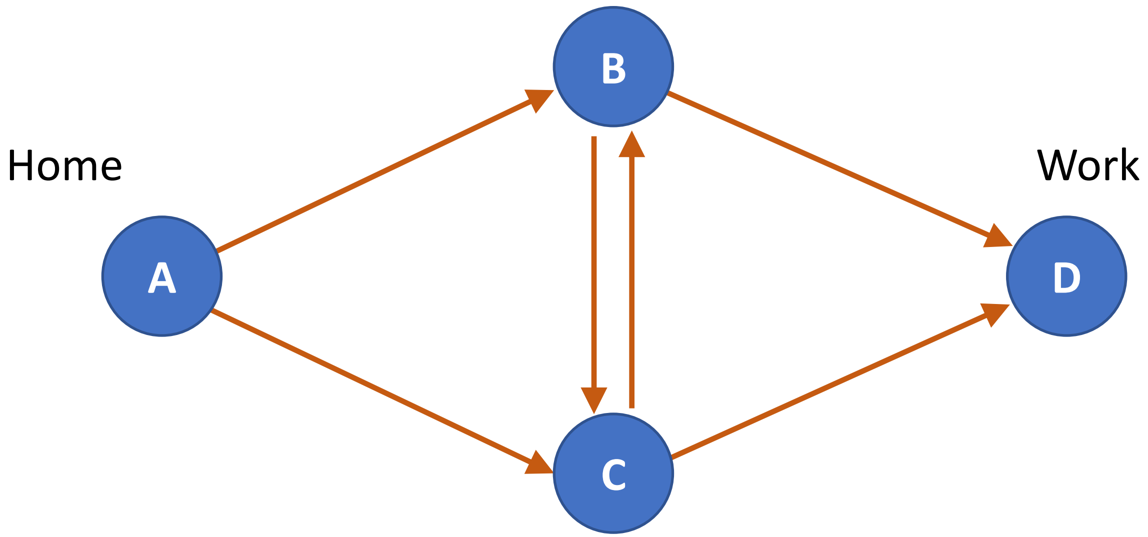


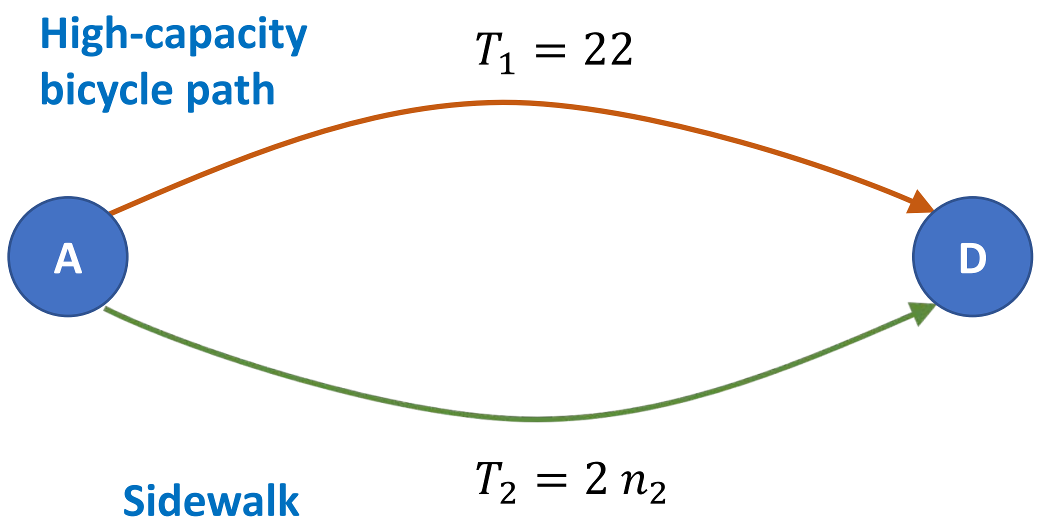
The solution space is symmetric with respect to swap of commuters (identical particles), so it is a system with many Nash equilibria.

We consider the problem in terms of populations, not individuals.

A simpler game: commuting with two paths

Two rules to this game:

- You can choose one of two paths to commute to school.
- There is a formula for each path to calculate the commute time.



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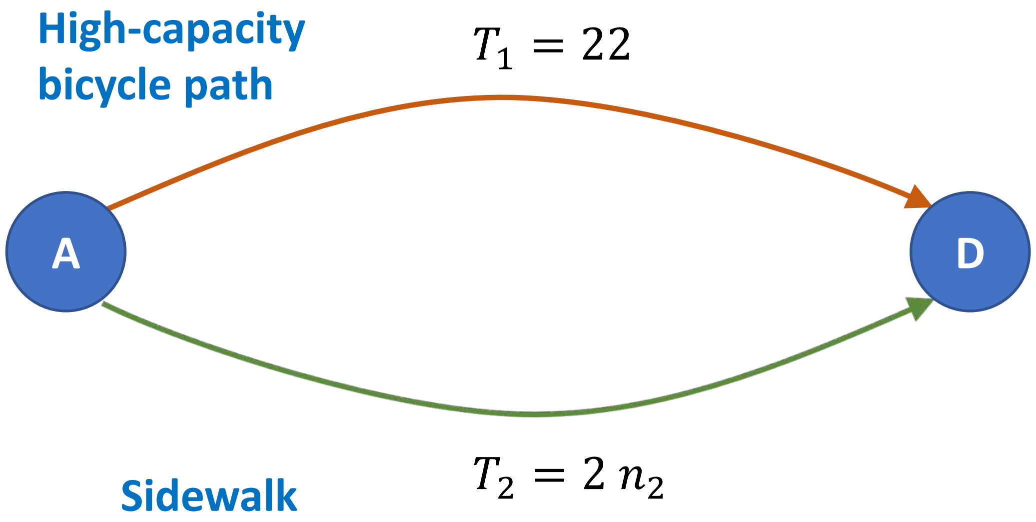

Imagine that we have a situation with $N = n_1 + n_2 = 20$.

For every person that travels path 2, the commute time increases by 2 minutes.

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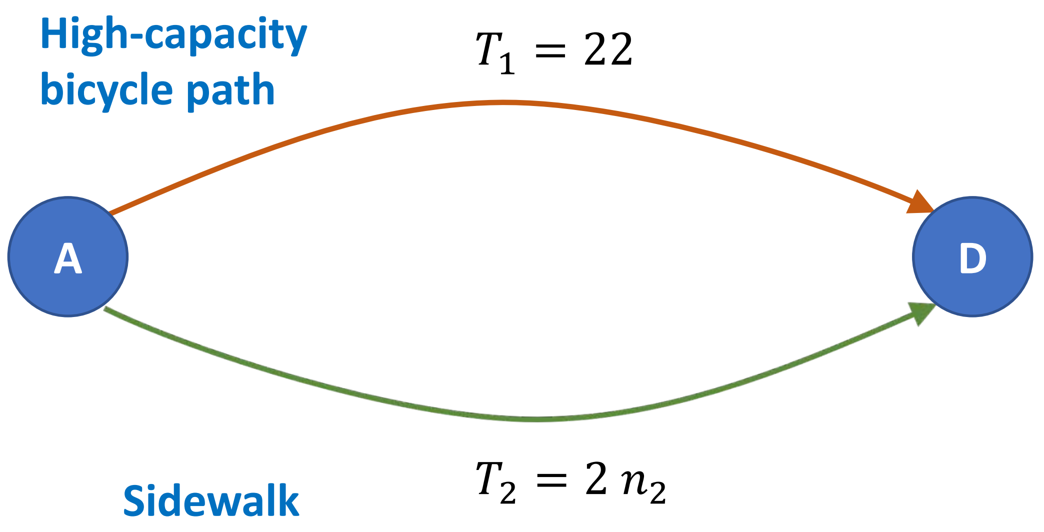

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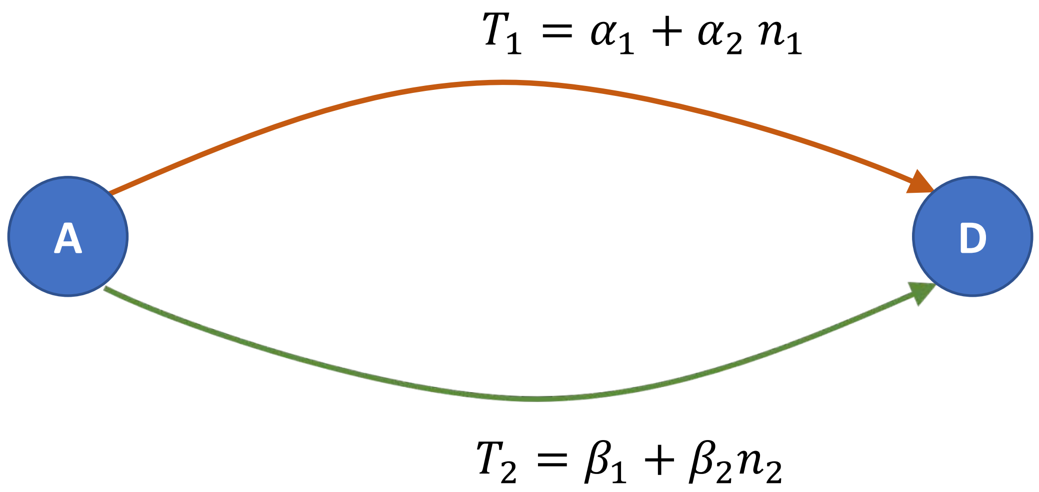
Nash:
Equilibrium will be established when there is no better choice, meaning $T_1=T_2$.

This happens when $n_1=9$ and $n_2=11$.

Generalizing the problem

- Let us generalize the equation for the travel time so each path has a constant and a population term.



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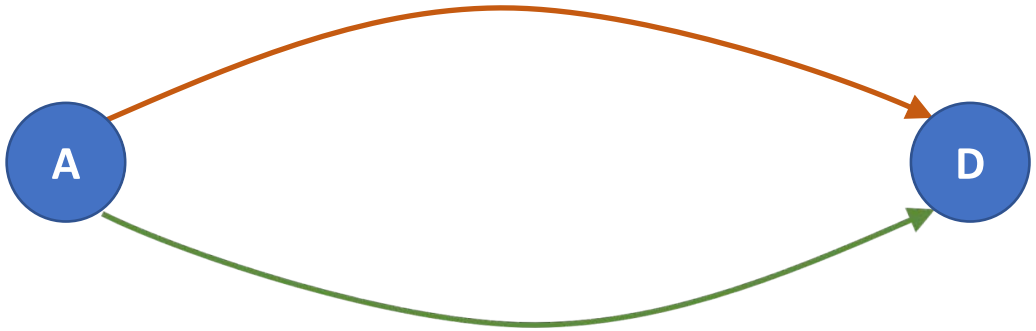
$T_1 = \alpha_1 + \alpha_2\, n_1$



$T_2 = \beta_1 + \beta_2 n_2$

3 equations for this system:

Road length/speed Congestion

$$T_1 = \boxed{\alpha_1} + \boxed{\alpha_2}\, n_1$$

$$T_2 = \boxed{\beta_1} + \boxed{\beta_2}\, n_2$$

$$N = n_1 + n_2$$

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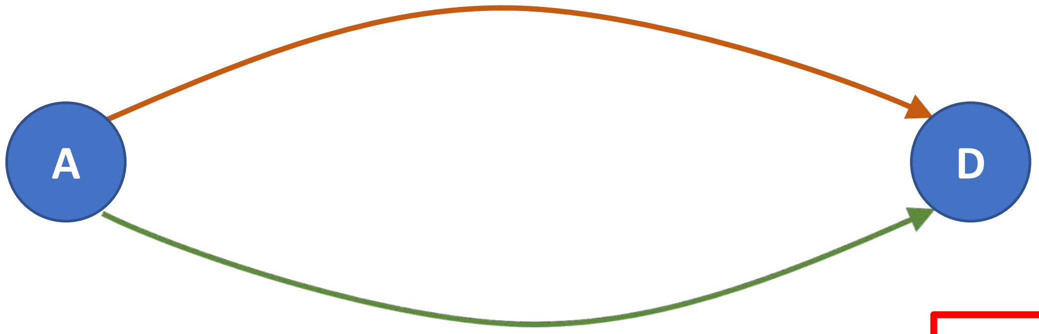
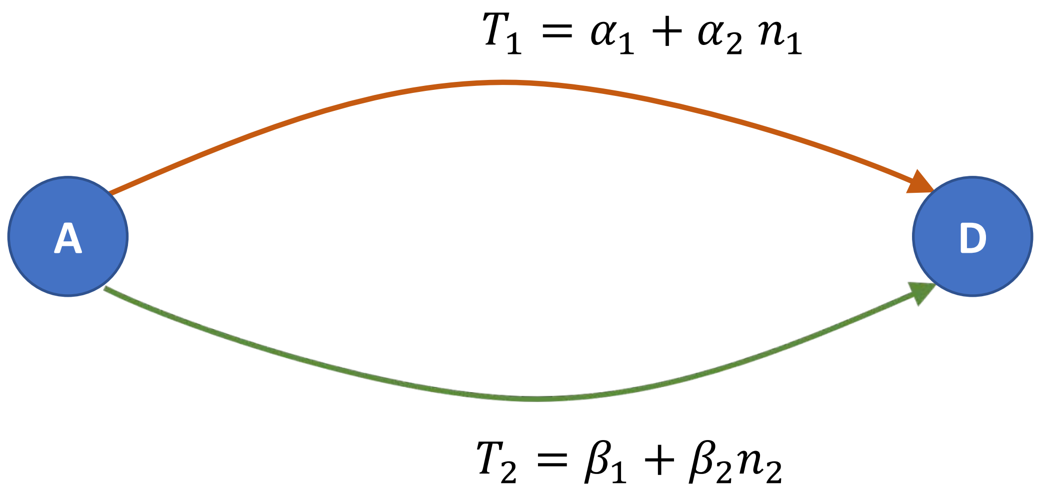


$T_2 = \beta_1 + \beta_2 n_2$

Better roads = shorter commute = smaller α_i, β_i

Solution by graphical analysis

Study the variation in commute times as a function of n_1:



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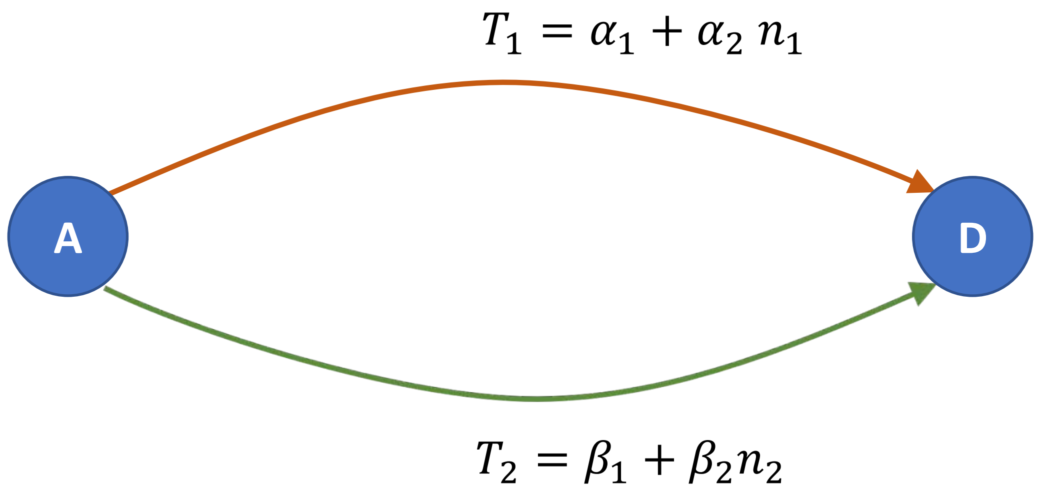

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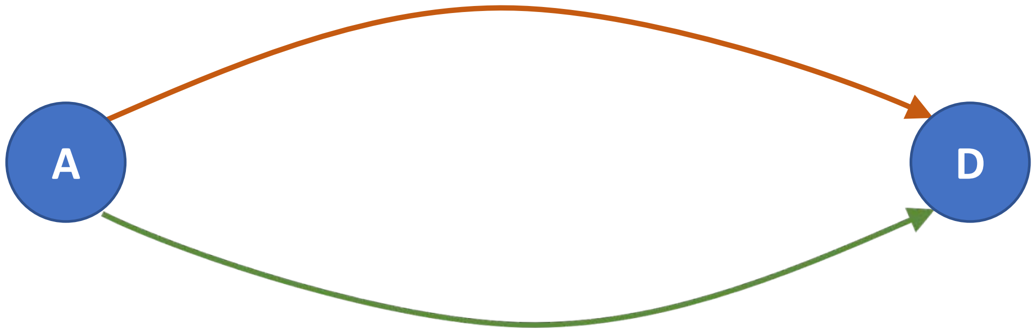

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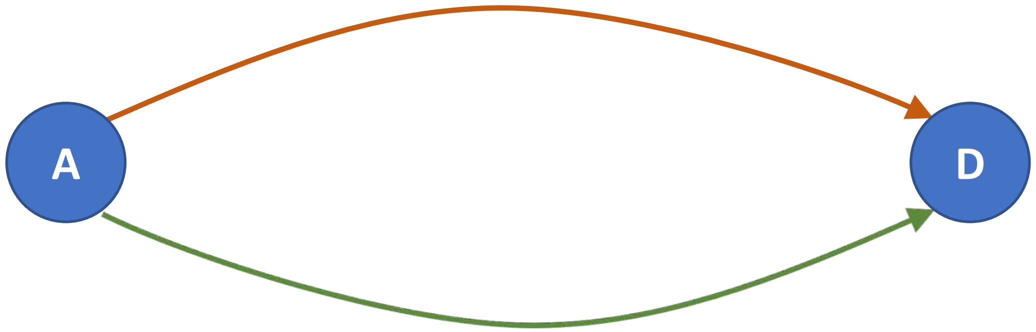
$$T_2 = \beta_1 + \beta_2\, n_2$$
$$= \beta_1 + \beta_2\, (N - n_1)$$
$$= (\beta_1 + \beta_2\, N) - \beta_2 n_1$$

$$N = n_1 + n_2$$

$$T_1 = \alpha_1 + \alpha_2\, n_1$$



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- A specific example:

$$N = n_1 + n_2 = 100$$

$$T_1 = 10 + 1.5\, n_1$$

$$T_2 = 80 - 0.5\, n_1$$

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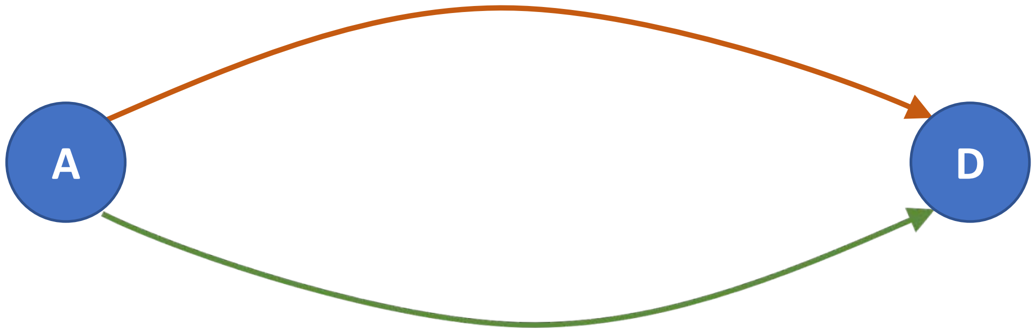

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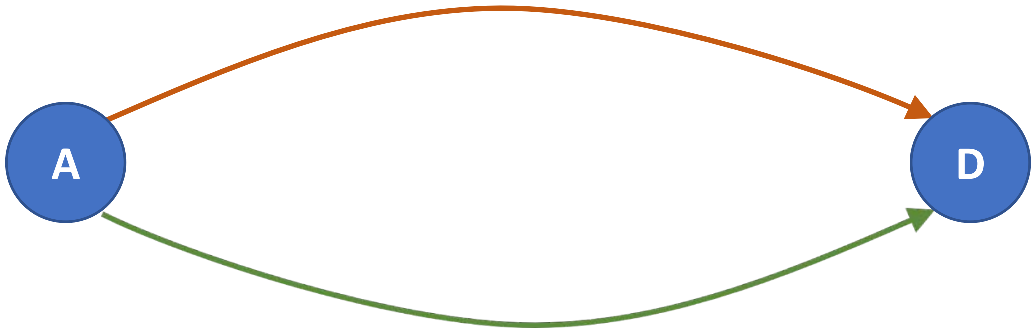

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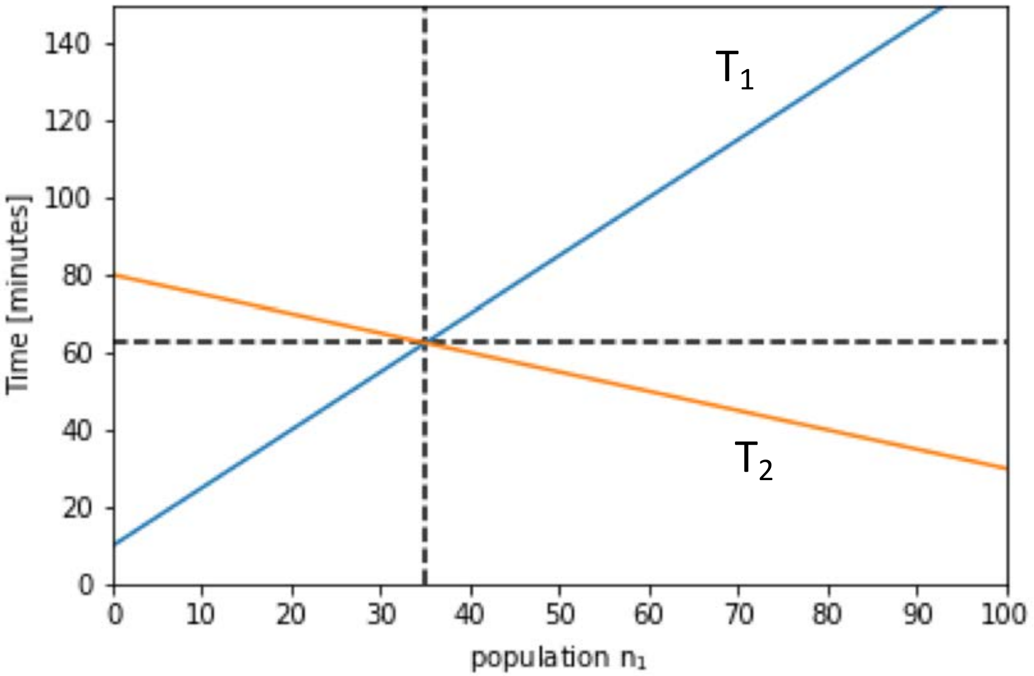

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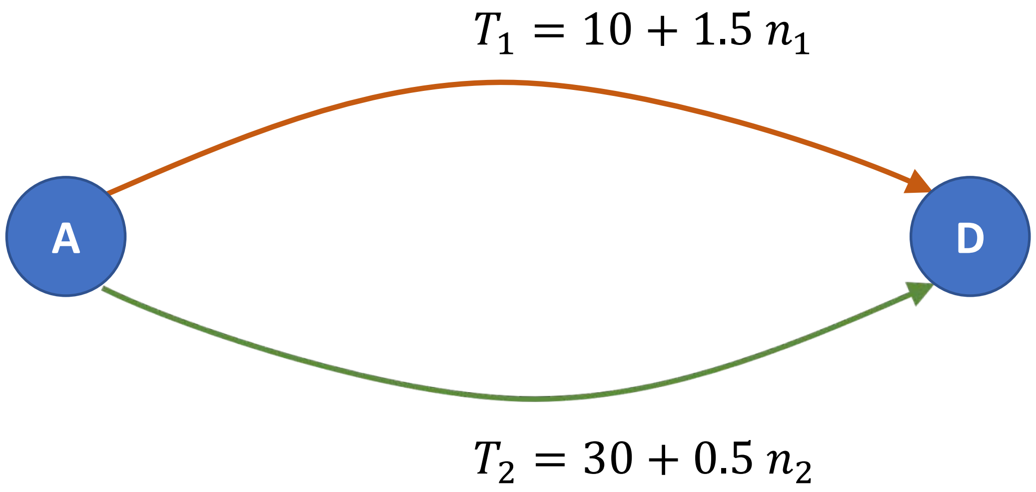
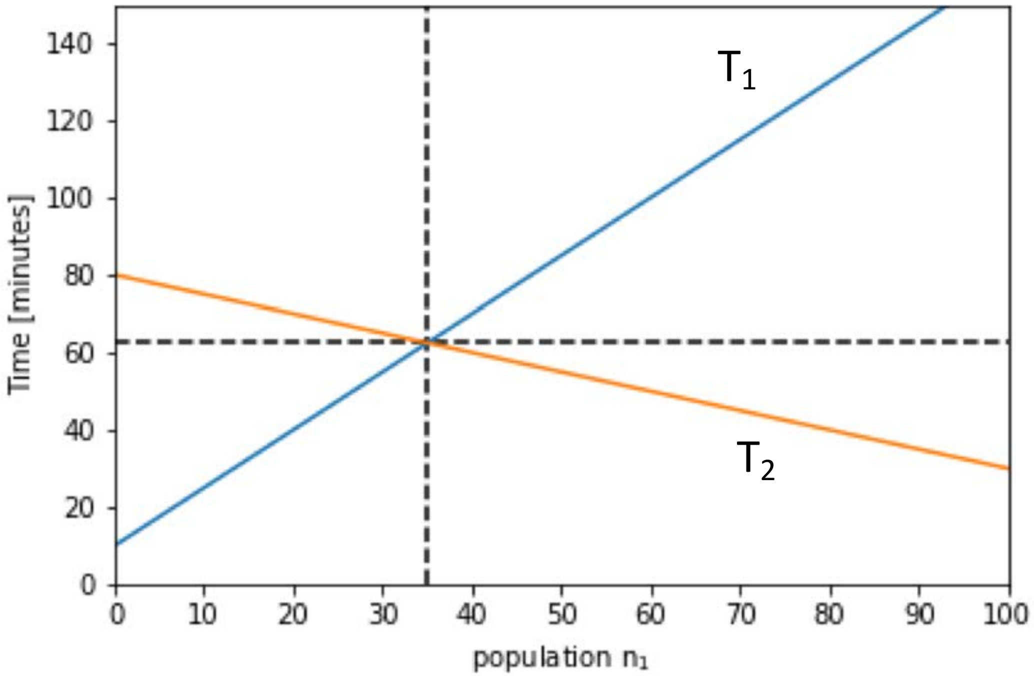


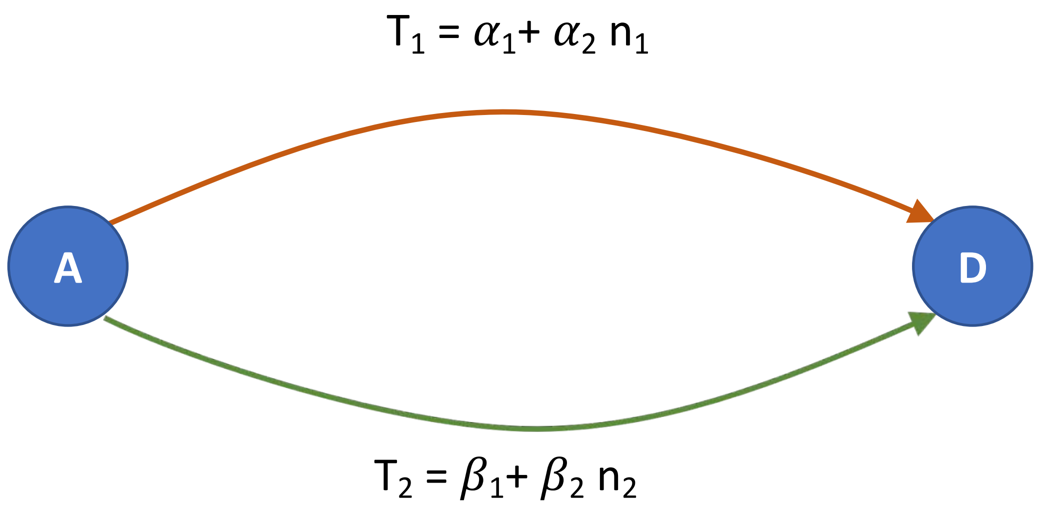
- A specific example:

$$N = n_1 + n_2 = 100 \qquad T = 62.5 \text{ minutes}$$
$$T_1 = 10 + 1.5\, n_1 \quad \Rightarrow \quad n_1 = 35$$
$$T_2 = 80 - 0.5\, n_1 \qquad n_2 = 65$$



Formalizing the method of solution

- We now seek a formal mathematical approach that can be generalized to higher dimensional problems.



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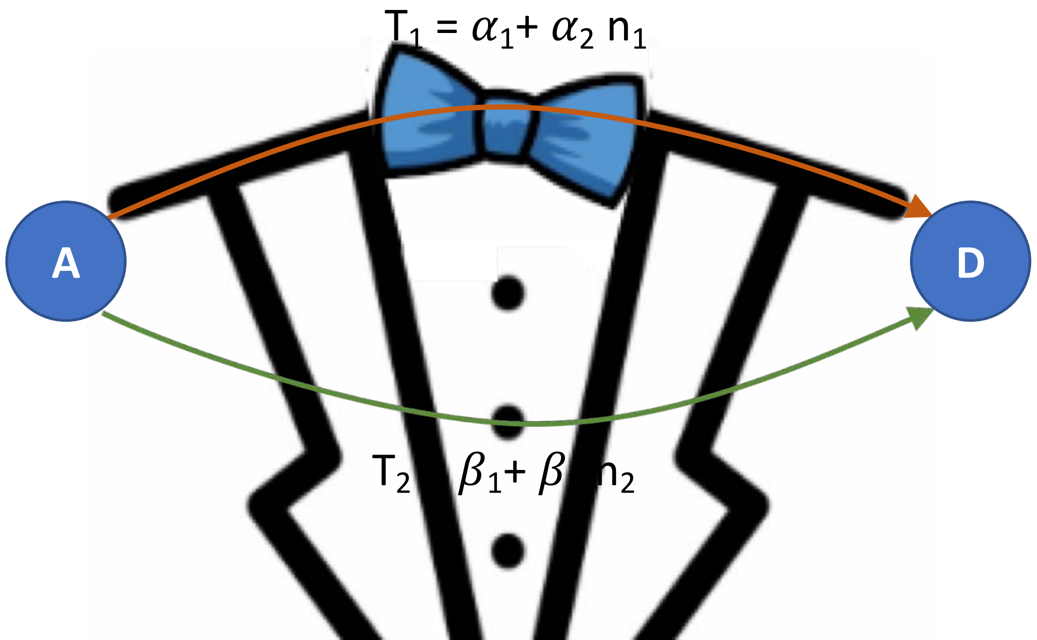

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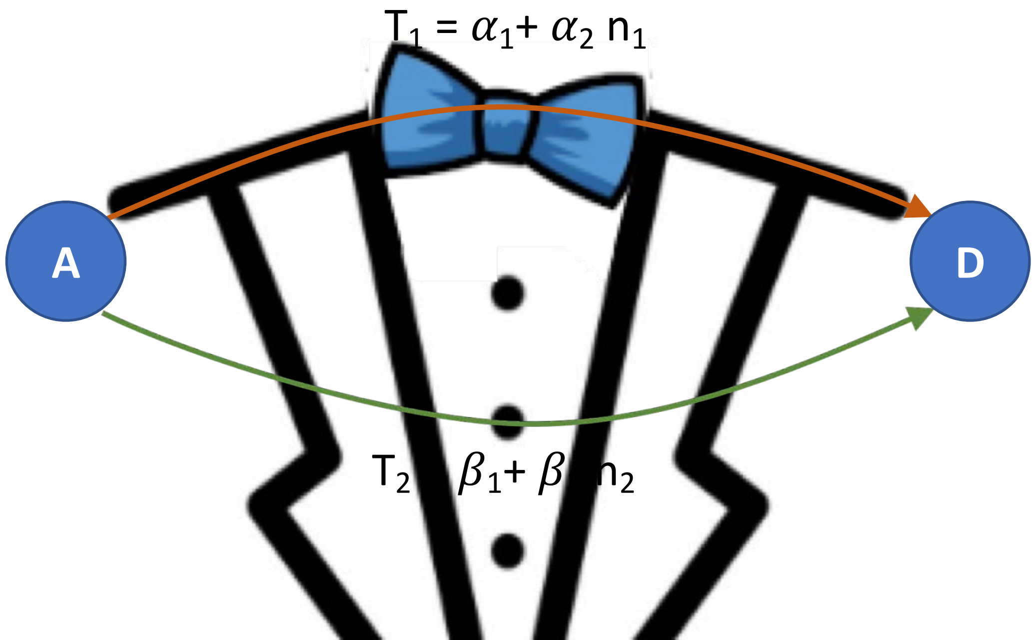

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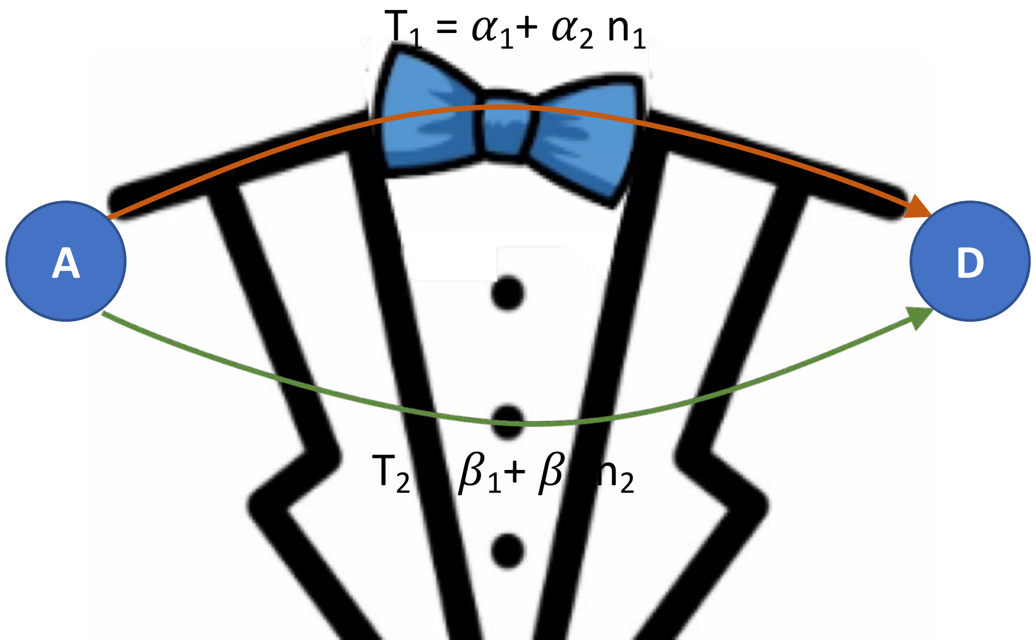

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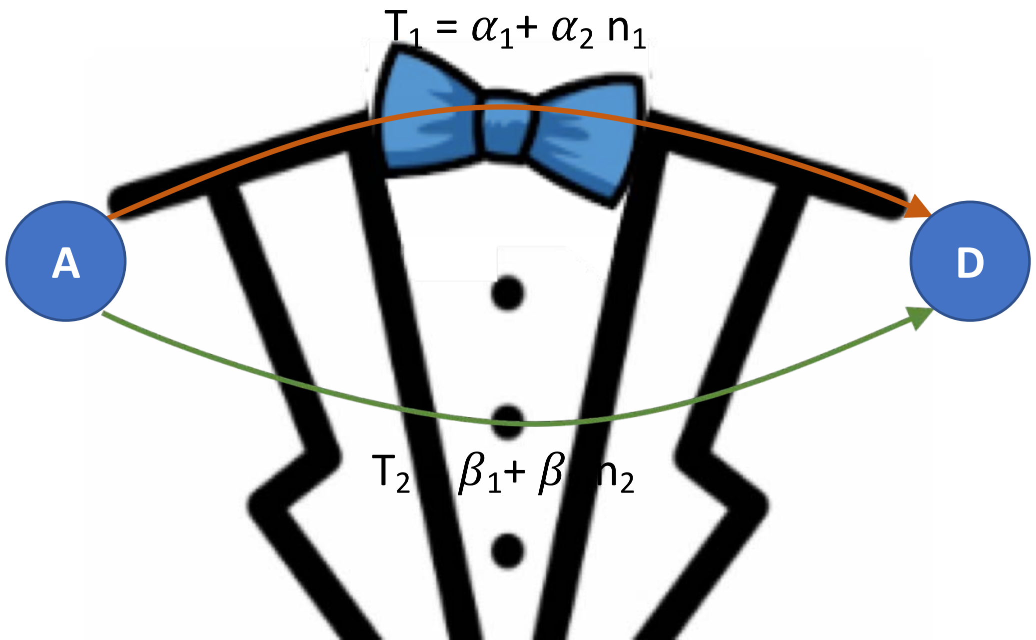

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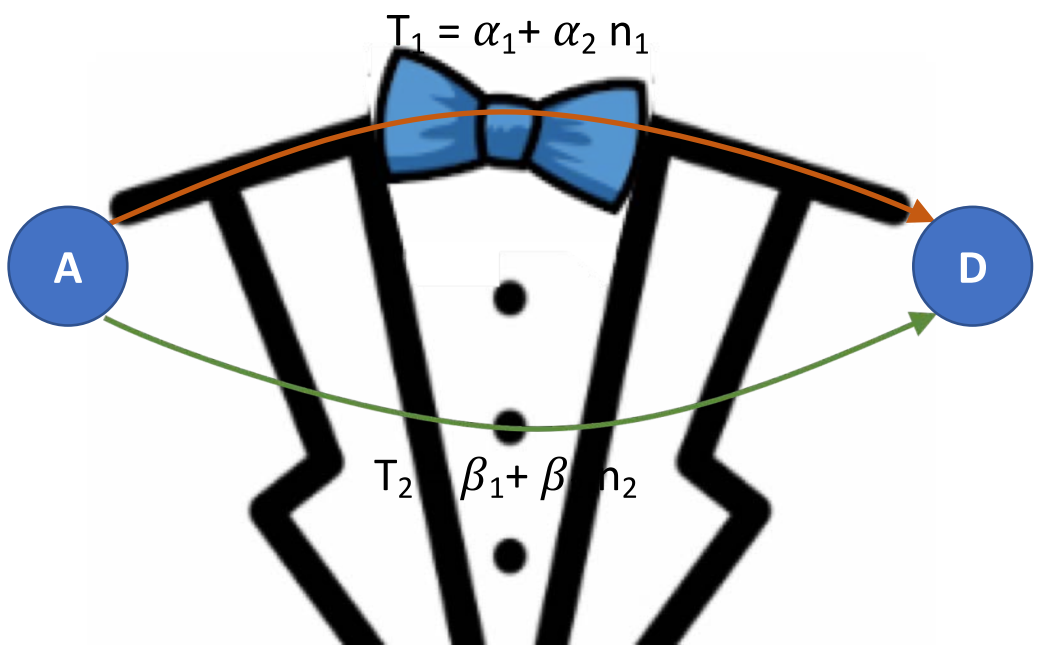

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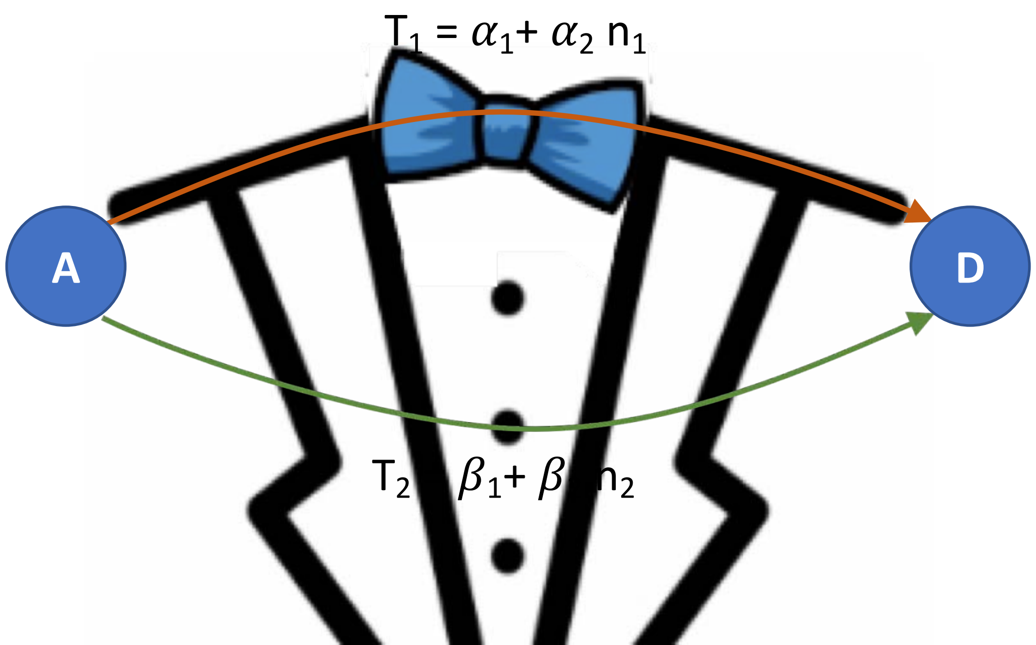

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$$\Downarrow$$

$$\begin{pmatrix} N \\ \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -\alpha_2 & 0 \\ 1 & 0 & -\beta_2 \end{pmatrix} \begin{pmatrix} T \\ n_1 \\ n_2 \end{pmatrix}$$

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We can solve for the unknowns by calculating the inverse matrix:

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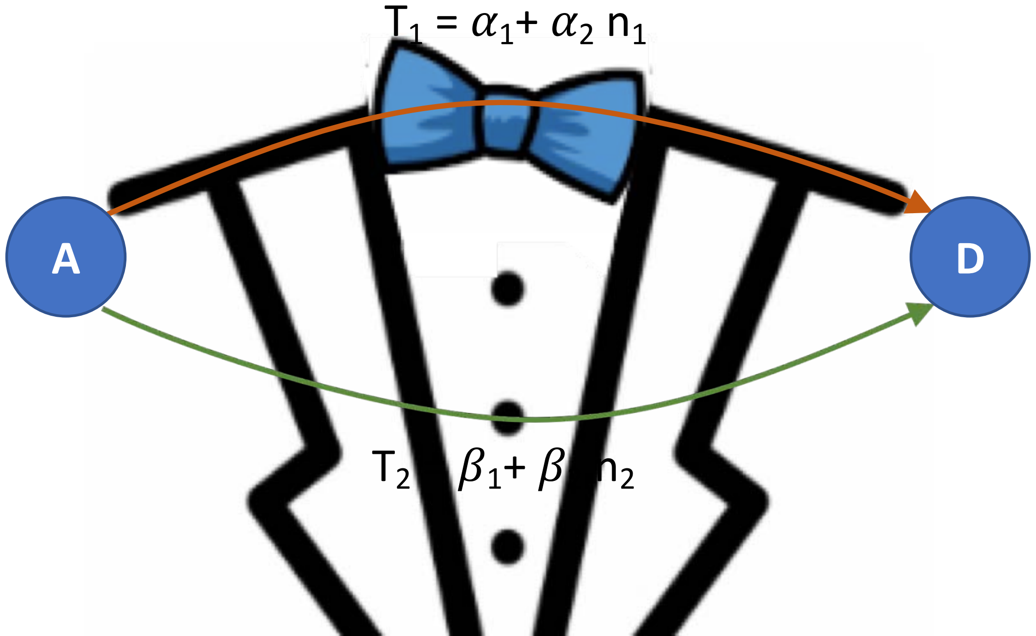

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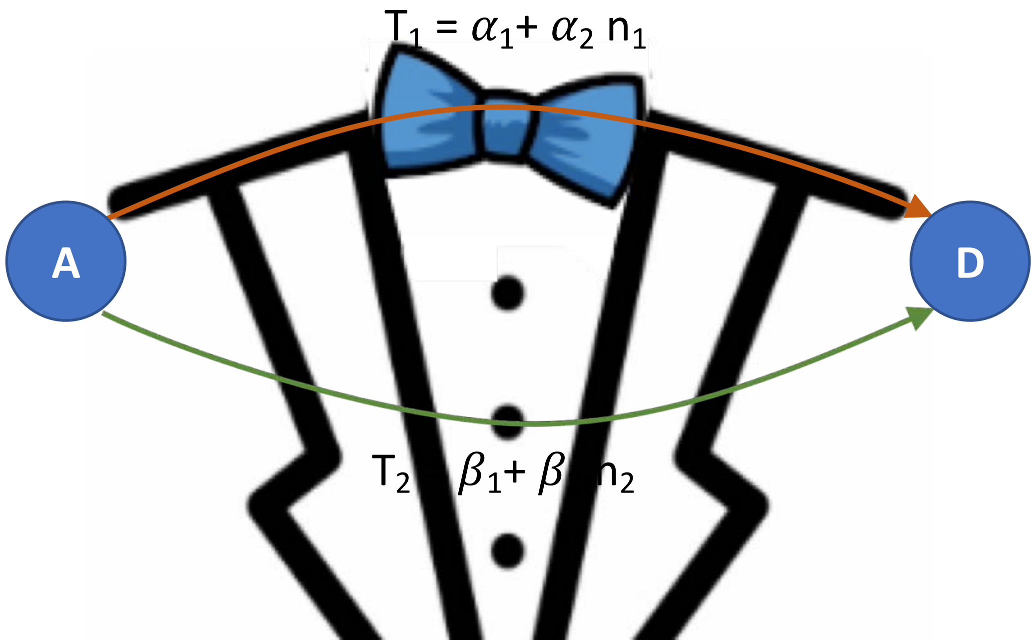

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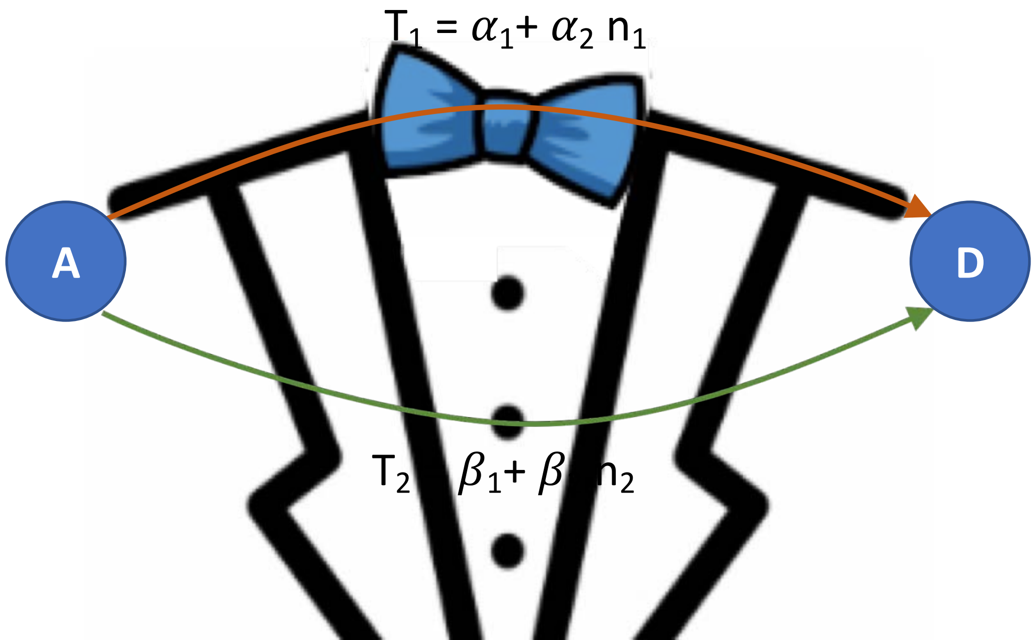

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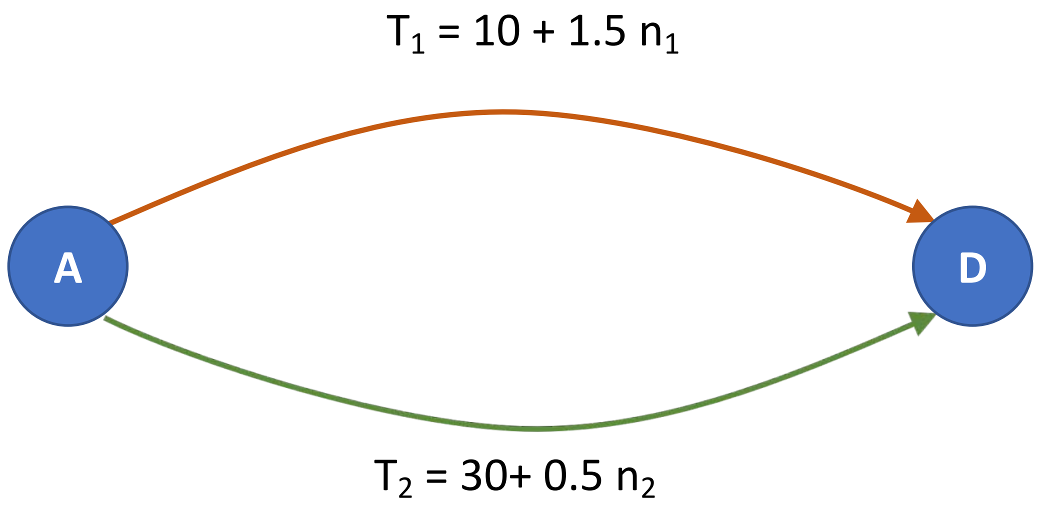
$\boxed{\text{Let's try it!}}$

Formalizing the method of solution

- Let's try the prior example with N=100.



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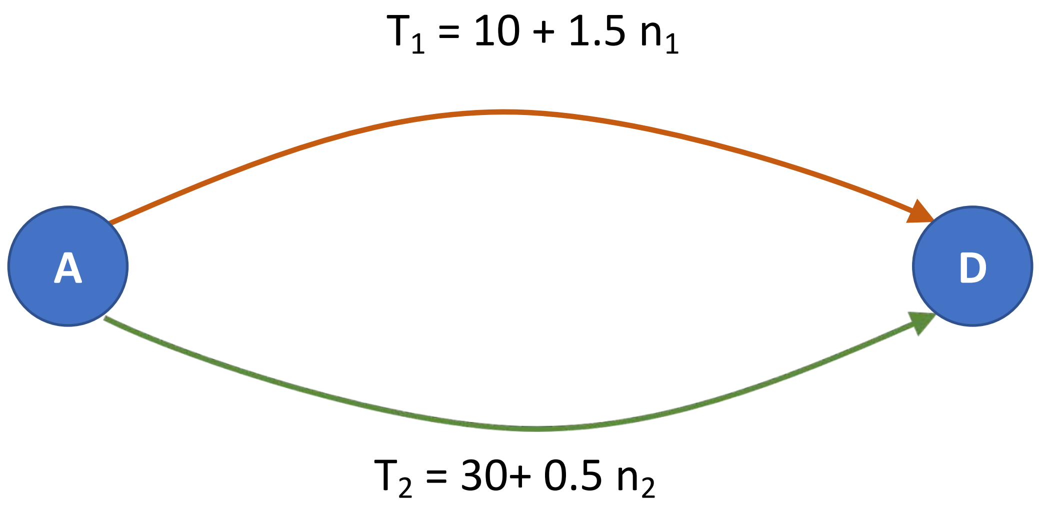
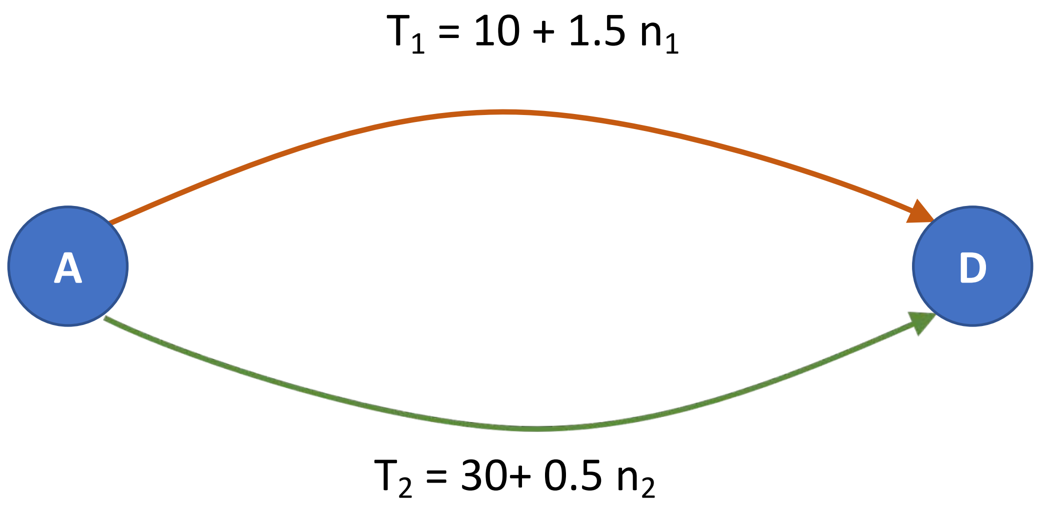


Formalizing the method of solution

- Let's try the prior example with N=100.

$$\begin{pmatrix} T \\ n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0.375 & 0.25 & 0.75 \\ 0.25 & -0.50 & 0.50 \\ 0.75 & 0.50 & -0.50 \end{pmatrix} \begin{pmatrix} 100 \\ 10 \\ 30 \end{pmatrix}$$



Formalizing the method of solution

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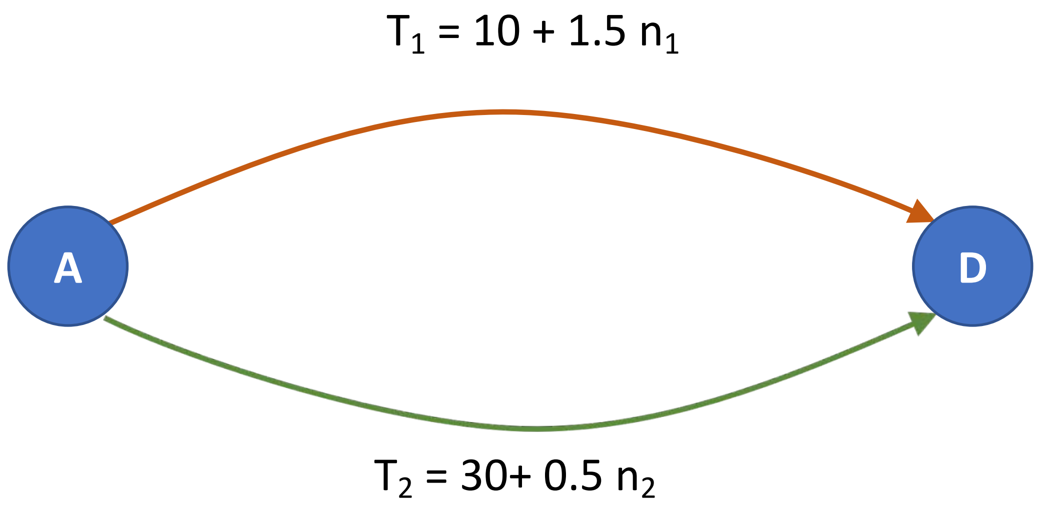
$$T = (0.375)(100) + (0.25)(10) + (0.75)(30)$$

$$= 62.5 \text{ minutes}$$



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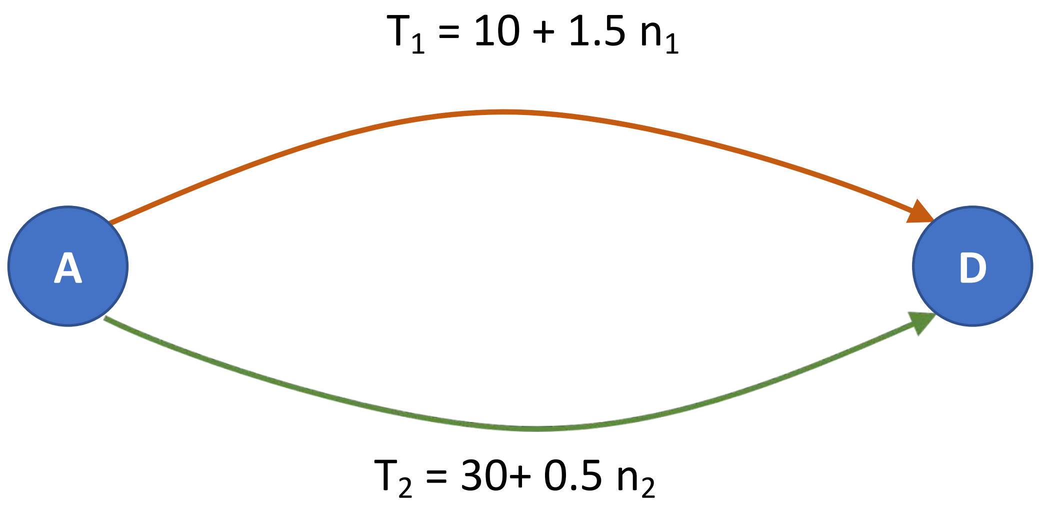

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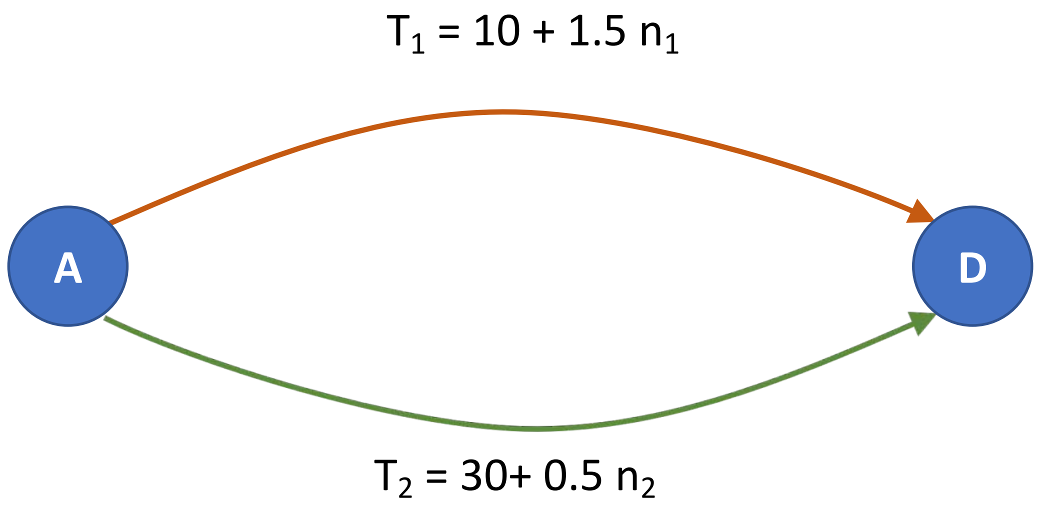
$$n_2 = (0.75)(100) + (0.50)(10) + (-0.50)(30)$$
$$= 65$$



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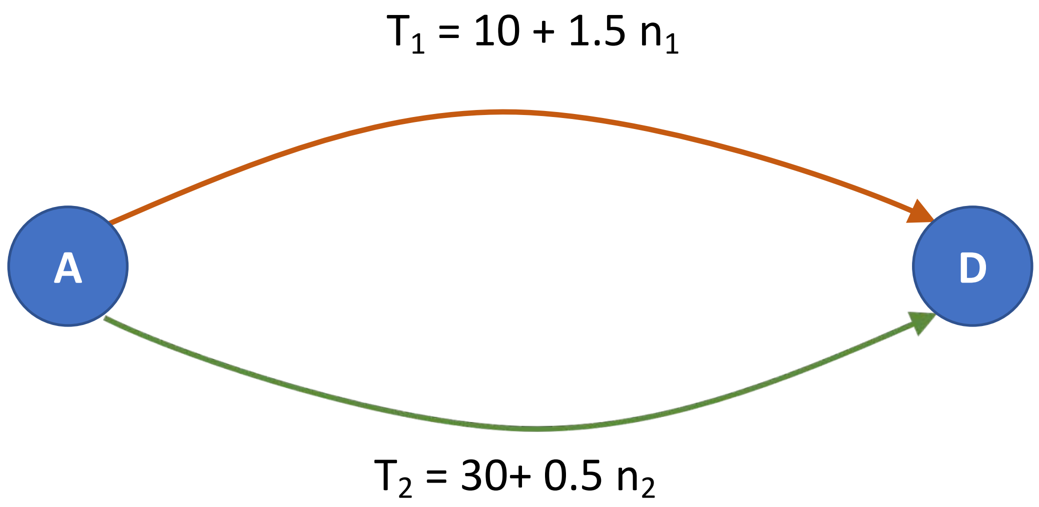

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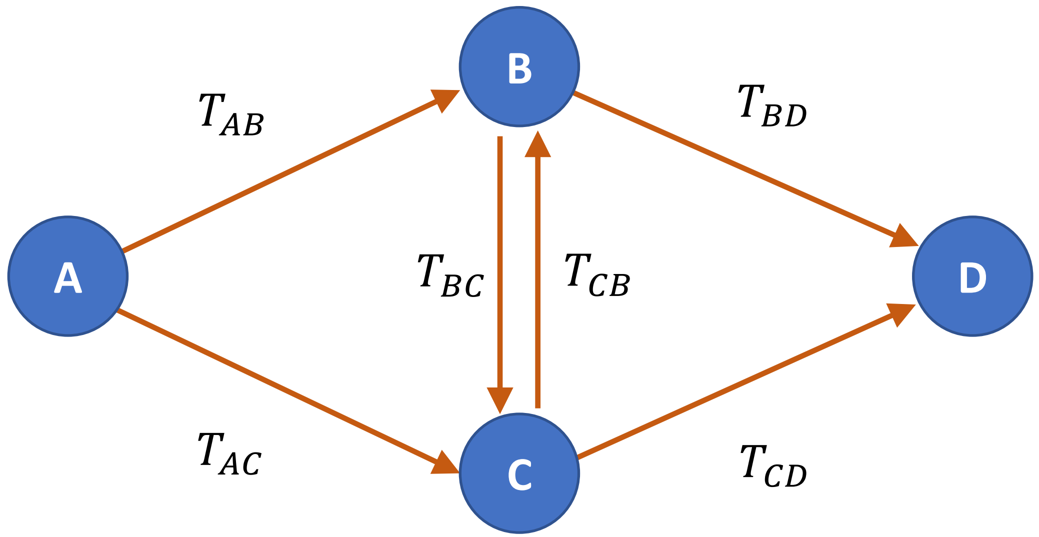
Level up!

Commuting from A to D with intermediate points

- Welcome to the 5th dimension!



Commuting from A to D with intermediate points

- Welcome to the 5th dimension!
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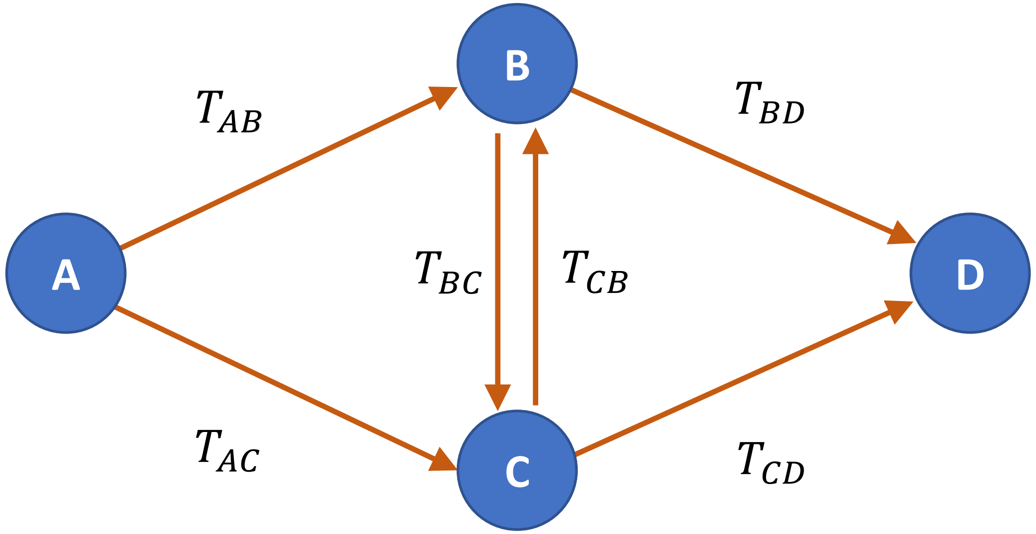

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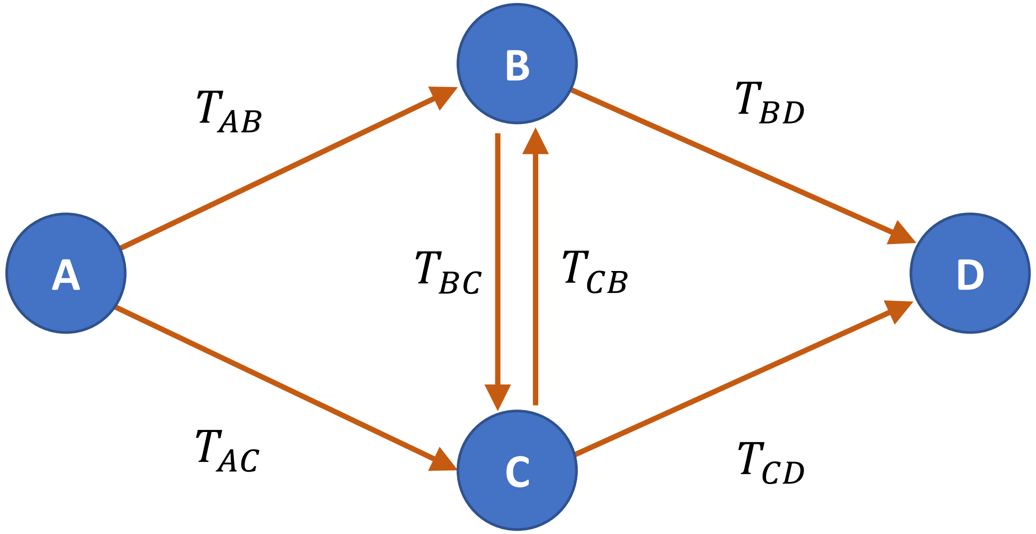

<u>Path segment times</u>

$$T_{AB} = \alpha_1 + \alpha_2\, n_{AB}$$

$$T_{BD} = \beta_1 + \beta_2\, n_{BD}$$

$$T_{AC} = \gamma_1 + \gamma_2\, n_{AC}$$

$$T_{CD} = \delta_1 + \delta_2\, n_{CD}$$

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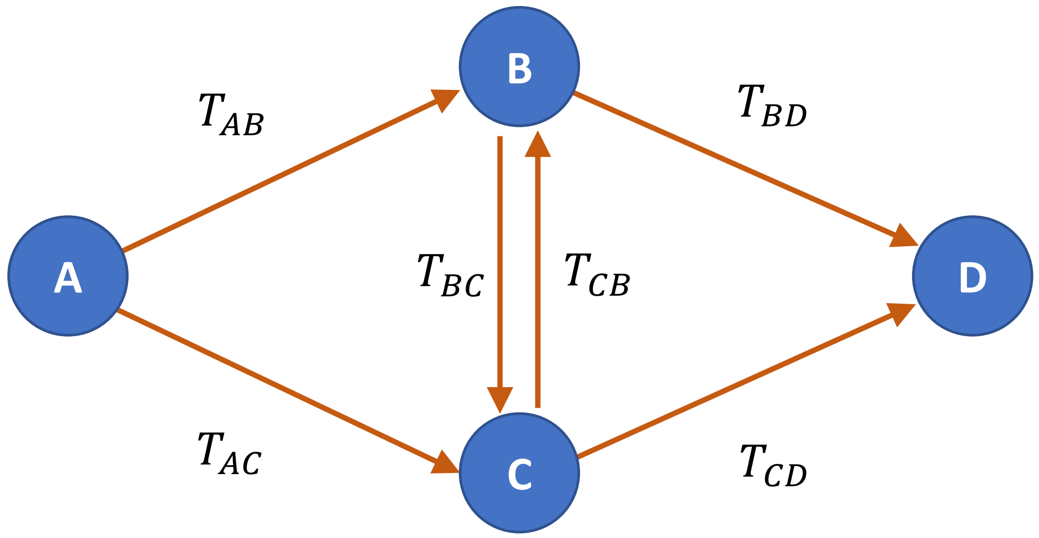

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Road length/speed

Congestion

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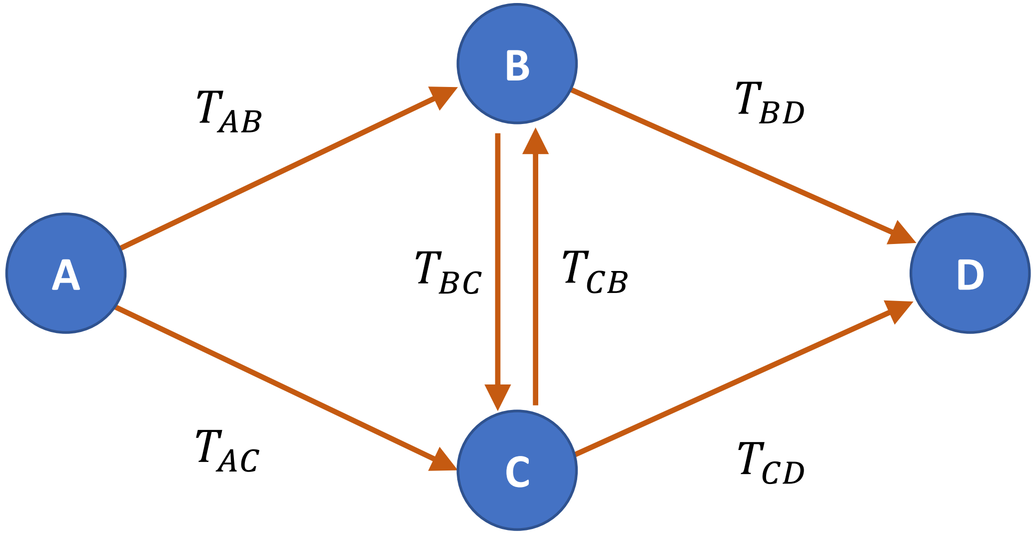

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Road length/speed

Congestion

<u>Junction Sums</u>

$$N = n_{AB} + n_{AC}$$
$$N = n_{BD} + n_{CD}$$
$$n_{AB} + n_{CB} = n_{BD} + n_{BC}$$
$$n_{AC} + n_{BC} = n_{CD} + n_{CB}$$

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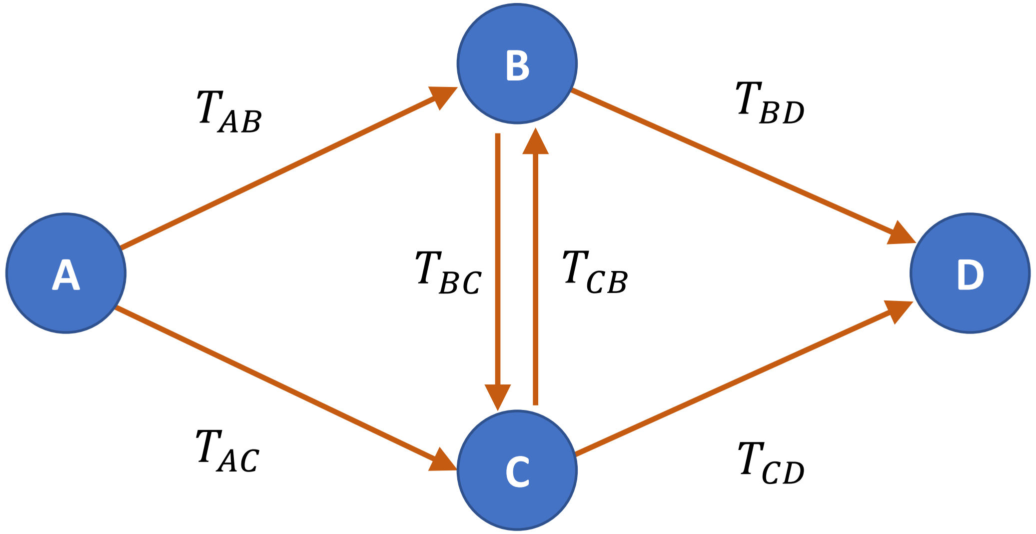

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Road length/speed Congestion

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Equilibrium

$$\begin{aligned} T_{AB} + T_{BD} &= T_{AC} + T_{CD} \\ &= T_{AB} + T_{BC} + T_{CD} \\ &= T_{AC} + T_{CB} + T_{BD} \end{aligned}$$

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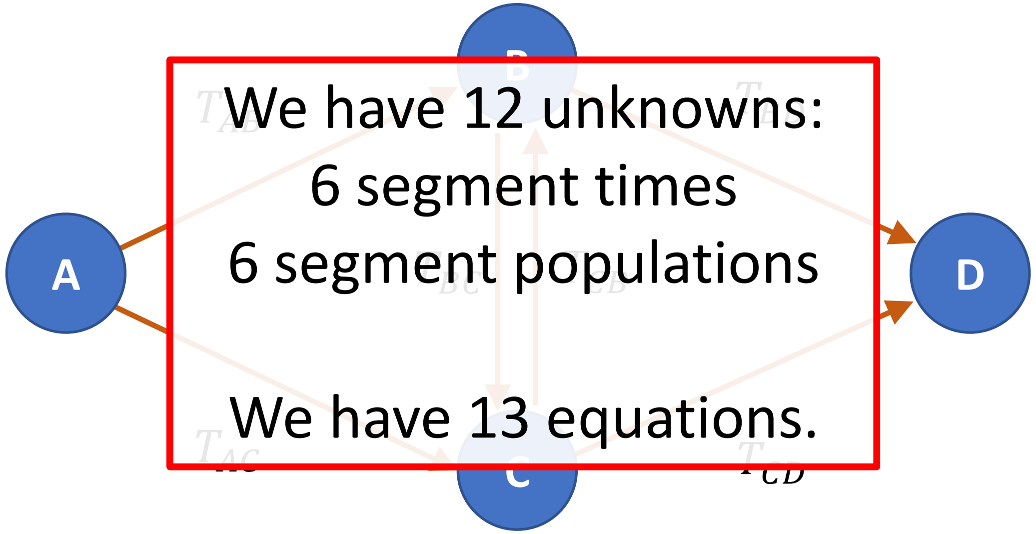

We have 12 unknowns:
6 segment times
6 segment populations

We have 13 equations.

<u>Path segment times</u>

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Road length/speed

Congestion

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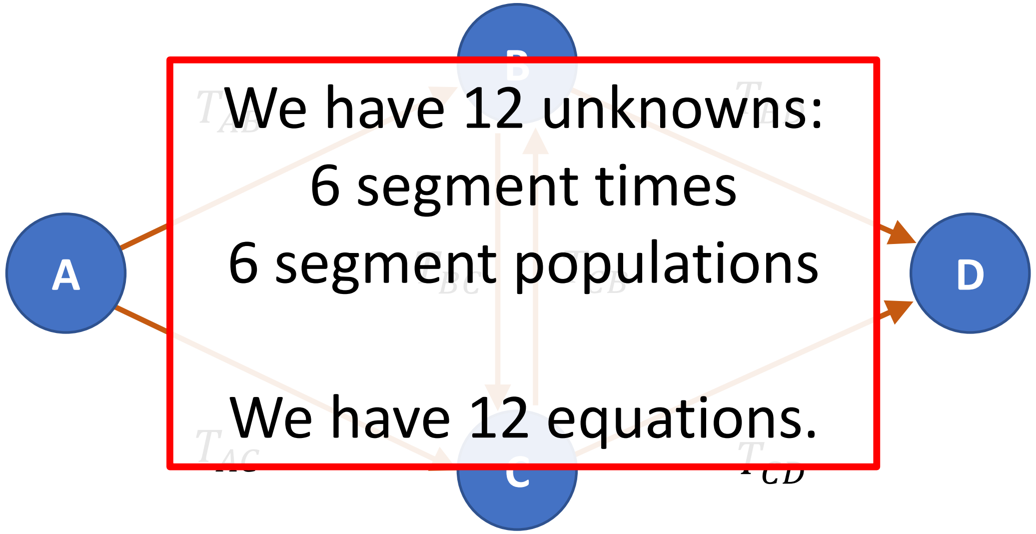

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Road length/speed

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$$\cancel{n_{AC} + n_{BC} = n_{CD} + n_{CB}}$$

Equilibrium

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$$= T_{AB} + T_{BC} + T_{CD}$$
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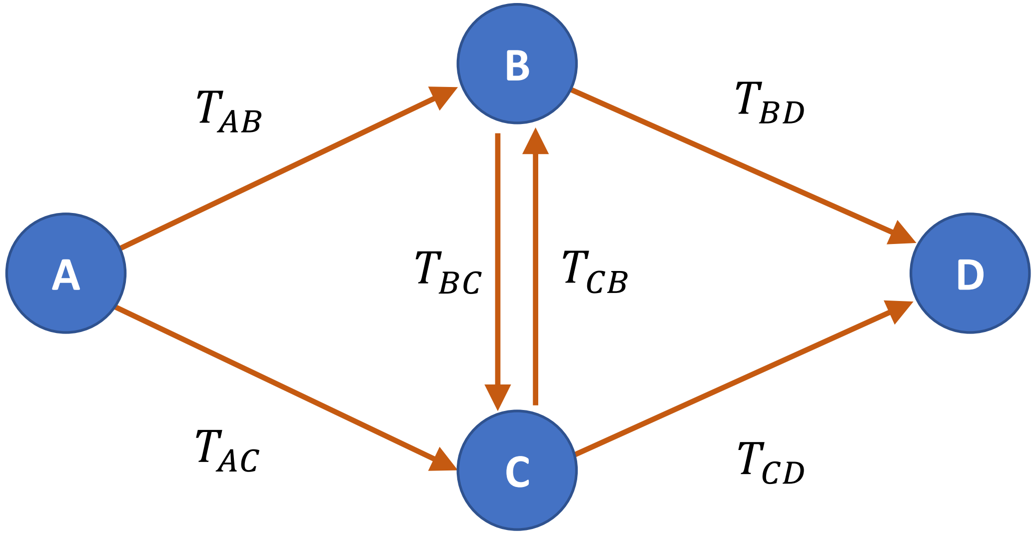
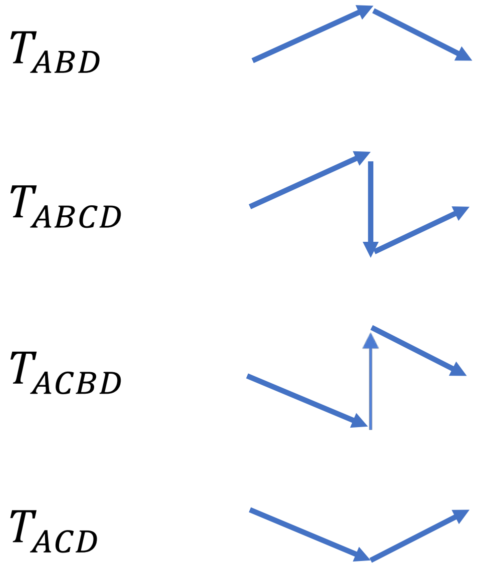
Commuting from A to D with intermediate points

- Welcome to the 5th dimension!
- Equilibrium occurs when all possible paths have equal commute time.
- There are four reasonable (non-looping) paths:





$$N = n_{ABD} + n_{ABCD} + n_{ACD} + n_{ACBD}$$

Commuting from A to D with intermediate points

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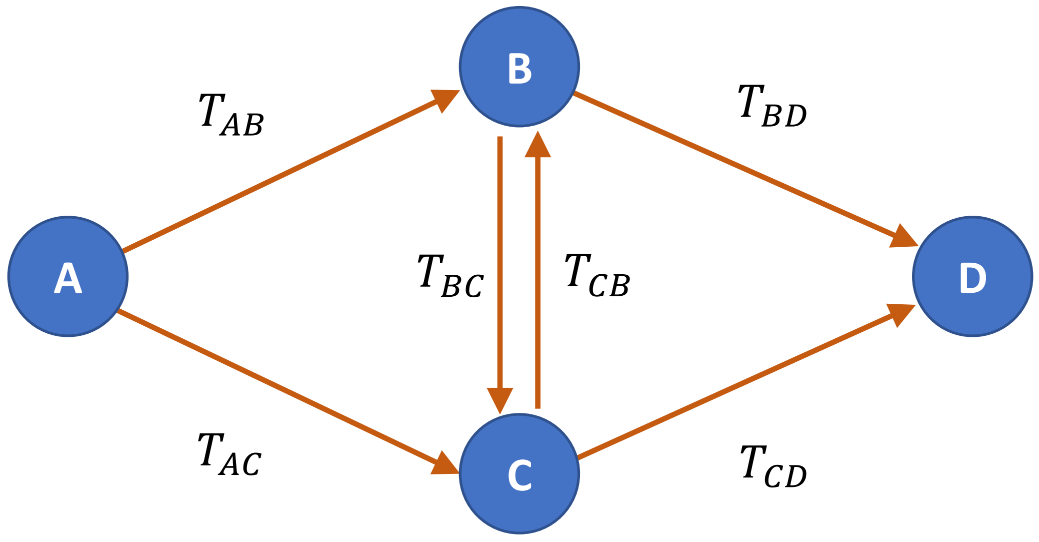

We have:
1 equilibrium time
+ 4 populations
= 5 unknowns

We have 5 equations.

Good to go.

Welcome to the matrix

$$\begin{pmatrix} N \\ \alpha_1 + \beta_1 \\ \alpha_1 + \delta_1 + \epsilon_1 \\ \beta_1 + \gamma_1 + \theta_1 \\ \gamma_1 + \delta_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & -(\alpha_2 + \beta_2) & -\alpha_2 & -\beta_2 & 0 \\ 1 & -\alpha_2 & -(\alpha_2 + \delta_2 + \epsilon_2) & 0 & -\delta_2 \\ 1 & -\beta_2 & 0 & -(\beta_2 + \gamma_2 + \theta_2) & -\gamma_2 \\ 1 & 0 & -\delta_2 & -\gamma_2 & -(\gamma_2 + \delta_2) \end{pmatrix} \begin{pmatrix} T \\ n_{ABD} \\ n_{ABCD} \\ n_{ACBD} \\ n_{ACD} \end{pmatrix}$$

Welcome to the matrix

$$
\begin{pmatrix} \boxed{N} \\ \alpha_1 + \beta_1 \\ \alpha_1 + \delta_1 + \epsilon_1 \\ \beta_1 + \gamma_1 + \theta_1 \\ \gamma_1 + \delta_1 \end{pmatrix}
=
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & -(\alpha_2 + \beta_2) & -\alpha_2 & -\beta_2 & 0 \\
1 & -\alpha_2 & -(\alpha_2 + \delta_2 + \epsilon_2) & 0 & -\delta_2 \\
1 & -\beta_2 & 0 & -(\beta_2 + \gamma_2 + \theta_2) & -\gamma_2 \\
1 & 0 & -\delta_2 & -\gamma_2 & -(\gamma_2 + \delta_2)
\end{pmatrix}
\begin{pmatrix} T \\ \boxed{n_{ABD}} \\ \boxed{n_{ABCD}} \\ \boxed{n_{ACBD}} \\ \boxed{n_{ACD}} \end{pmatrix}
$$

Welcome to the matrix

$$\begin{pmatrix} N \\ \alpha_1+\beta_1 \\ \alpha_1+\delta_1+\epsilon_1 \\ \beta_1+\gamma_1+\theta_1 \\ \gamma_1+\delta_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & -(\alpha_2+\beta_2) & -\alpha_2 & -\beta_2 & 0 \\ 1 & -\alpha_2 & -(\alpha_2+\delta_2+\epsilon_2) & 0 & -\delta_2 \\ 1 & -\beta_2 & 0 & -(\beta_2+\gamma_2+\theta_2) & -\gamma_2 \\ 1 & 0 & -\delta_2 & -\gamma_2 & -(\gamma_2+\delta_2) \end{pmatrix} \begin{pmatrix} T \\ n_{ABD} \\ n_{ABCD} \\ n_{ACBD} \\ n_{ACD} \end{pmatrix}$$

Apply standard matrix inversion techniques:

$$\begin{pmatrix} T \\ n_{ABD} \\ n_{ABCD} \\ n_{ACBD} \\ n_{ACD} \end{pmatrix} = \begin{pmatrix}
\frac{b(cef+def+cd(e+f))+a(cef+def+cd(e+f)+b(c+d)(e+f))}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{bce+cfe+d(a+e)f+cd(e+f)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{(bc-ad)f}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{ade-bce}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{a(d+e)f+be(c+f)+ab(e+f)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} \\
\frac{bce+cfe+d(a+e)f+cd(e+f)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & -\frac{(a+c+e)(b+d+f)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{(a+c)(b+d+f)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{(b+d)(a+c+e)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & -\frac{bc+dc+a(b+d)-ef}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} \\
\frac{(bc-ad)f}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{(a+c)(b+d+f)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & -\frac{cd+fd+cf+b(c+f)+a(b+d+f)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & -\frac{(a+c)(b+d)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{(b+d)(a+c+f)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} \\
\frac{ade-bce}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{(b+d)(a+c+e)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & -\frac{(a+c)(b+d)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & -\frac{cd+ed+ce+b(c+e)+a(b+d+e)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{(a+c)(b+d+e)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} \\
\frac{a(d+e)f+be(c+f)+ab(e+f)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & -\frac{bc+dc+a(b+d)-ef}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{(b+d)(a+c+f)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & \frac{(a+c)(b+d+e)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)} & -\frac{(b+d+e)(a+c+f)}{cde+afe+bfe+cfe+dfe+cdf+ab(e+f)+bc(e+f)+ad(e+f)}
\end{pmatrix} \begin{pmatrix} N \\ \alpha_1+\beta_1 \\ \alpha_1+\delta_1+\epsilon_1 \\ \beta_1+\gamma_1+\theta_1 \\ \gamma_1+\delta_1 \end{pmatrix}$$

* This calculation was done with Wolfram Alpha.

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- In a sense yes, but we can start to inquire about the properties of the solution: how does the system behave under x, y or z conditions?

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- The question we are going to address now is how does the commute time vary as we change the quality of the connector road?

$$T_{BC} = \epsilon_1 + \epsilon_2 \, n_{BC}$$

Better connector road = smaller ϵ_2

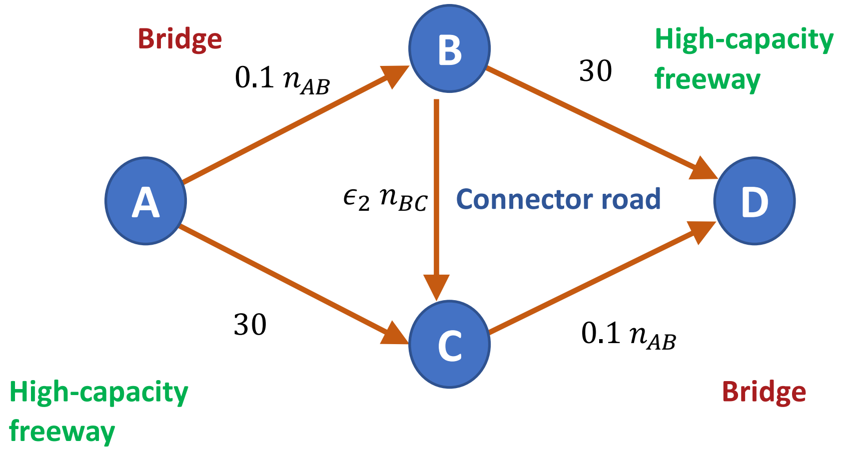
Road length is fixed.

Can decrease congestion by road improvements.

The Classic Braess example

$N = 200 \quad \alpha_1 = 0.0 \quad \beta_1 = 30 \quad \gamma_1 = 30 \quad \delta_1 = 0.0 \quad \epsilon_1 = 0$

$\alpha_2 = 0.1 \quad \beta_2 = 0.0 \quad \gamma_2 = 0.0 \quad \delta_2 = 0.1$



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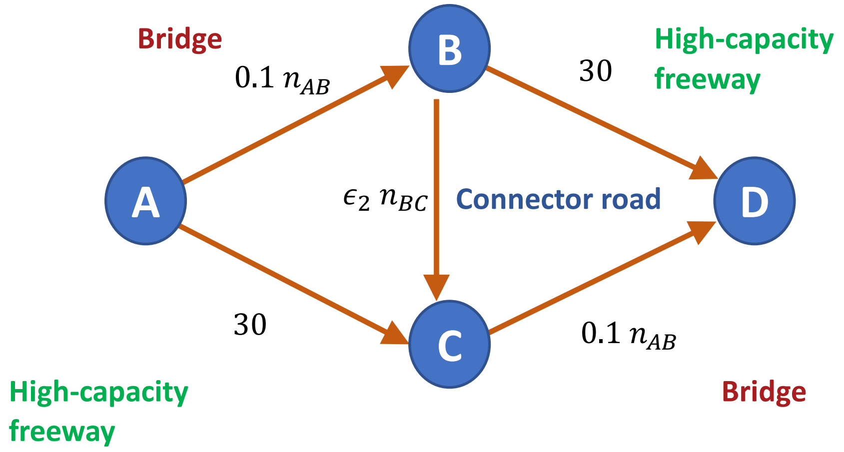


$$N = 200 \quad \alpha_1 = 0.0 \quad \beta_1 = 30 \quad \gamma_1 = 30 \quad \delta_1 = 0.0 \quad \epsilon_1 = 0$$

$$\alpha_2 = 0.1 \quad \beta_2 = 0.0 \quad \gamma_2 = 0.0 \quad \delta_2 = 0.1$$

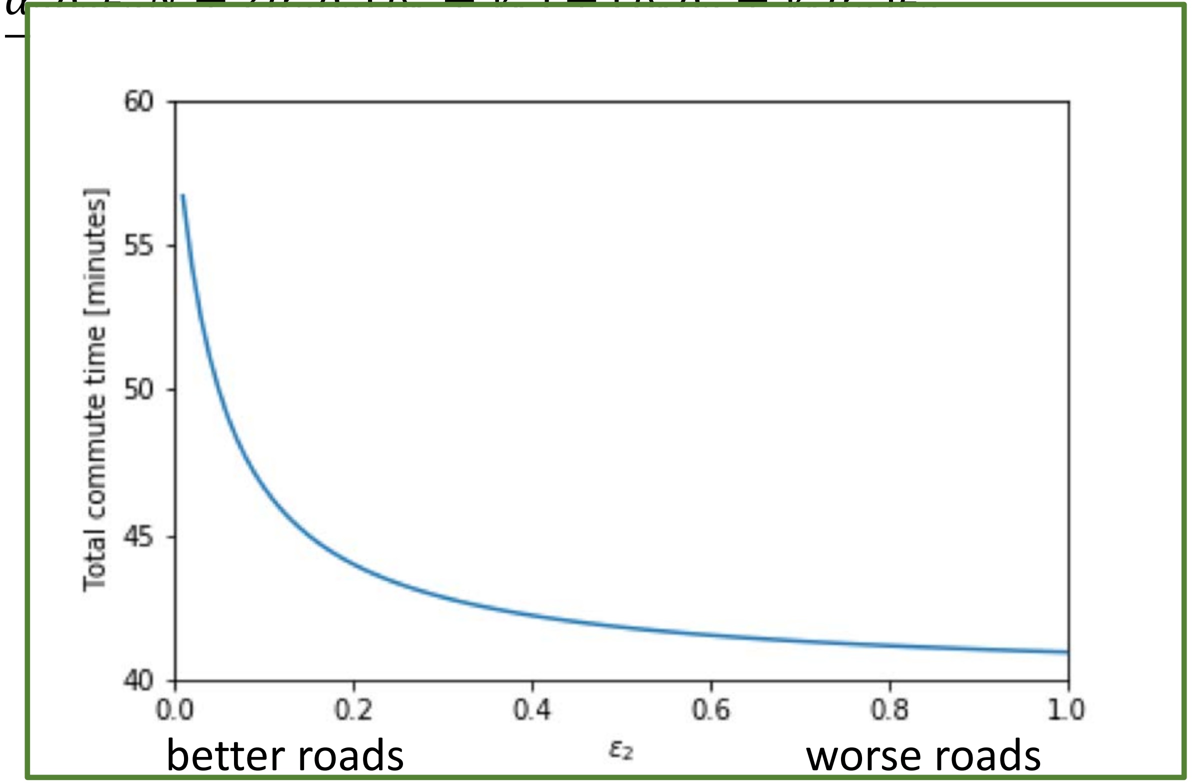
$$T = \frac{\alpha_2\delta_2\epsilon_2 N + 2\alpha_2\delta_2(\beta_1 + \gamma_1) + (\beta_1\delta_2 + \gamma_1\alpha_2)\epsilon_2}{\alpha_2\delta_2 + (\alpha_2 + \delta_2)\epsilon_2}$$

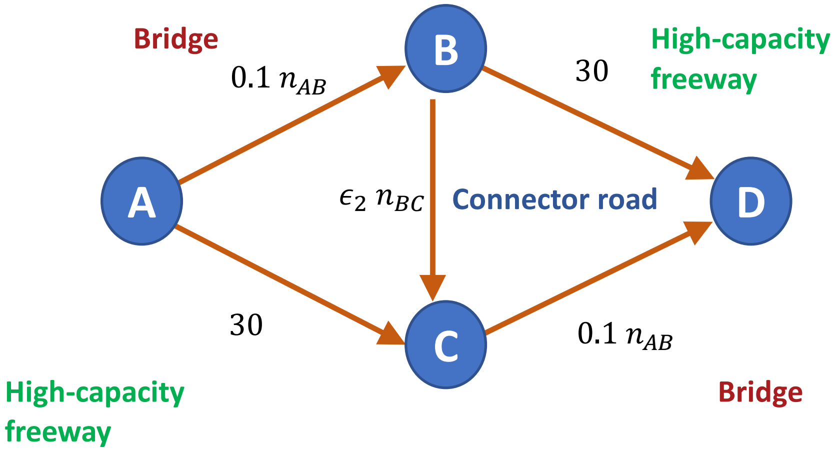
The Classic Braess example

$N = 200 \quad \alpha_1 = 0.0 \quad \beta_1 = 30 \quad \gamma_1 = 30 \quad \delta_1 = 0.0 \quad \epsilon_1 = 0$

$\alpha_2 = 0.1 \quad \beta_2 = 0.0 \quad \gamma_2 = 0.0 \quad \delta_2 = 0.1$

$T = \frac{\alpha\,\delta\,\epsilon\,N + 2\alpha\,\delta\,(\beta + \gamma) + (\beta\,\delta + \gamma\,\alpha)\epsilon}{\ldots}$





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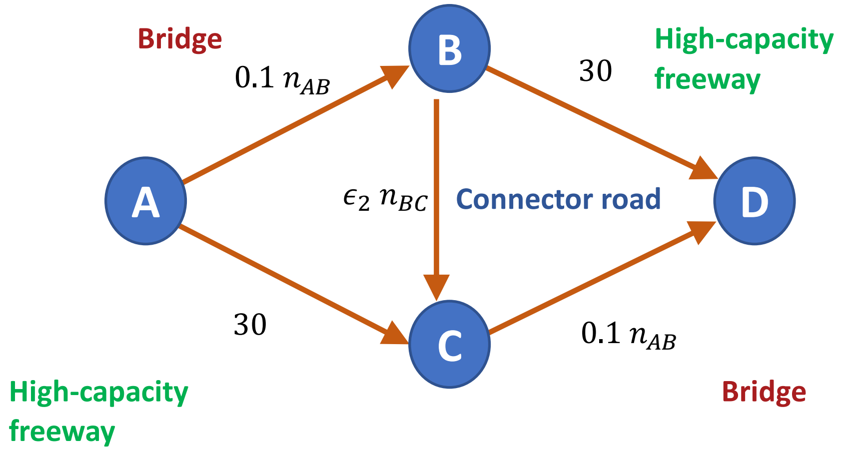

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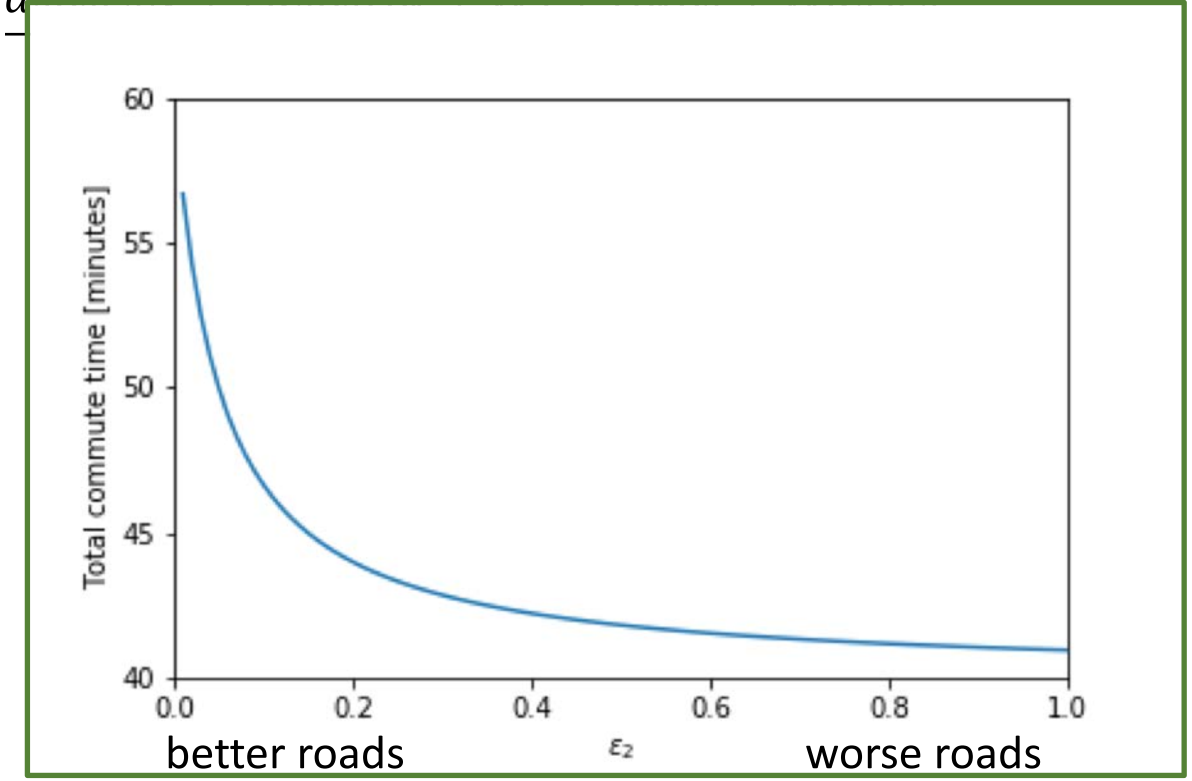

A decrease in congestion on the connector road (better roads) means longer commute times.

What?

Braess’ paradox:

- From Dietrich Braess (1968) in simulations of traffic flow.
    - Commute times can be lengthened by addition of new roads.
    - Conversely, removal of roads can decrease commute time.

Related topic: Why traffic apps make congestion worse
news.Berkeley.edu/story_jump/why-traffic-apps-make-congestion-worse

Long roads with congestion

$N = 200 \quad \alpha_1 = 30 \quad \beta_1 = 10 \quad \gamma_1 = 10 \quad \delta_1 = 30 \quad \epsilon_1 = 10$

$\alpha_2 = 0.1 \quad \beta_2 = 0.5 \quad \gamma_2 = 0.5 \quad \delta_2 = 0.1$

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$$T = \frac{2\alpha_2\gamma_2(\beta_2+\delta_2) + 2(\alpha_2+\gamma_2)\beta_2\delta_2 + (\alpha_2+\beta_2)(\gamma_2+\delta_2)\epsilon_2}{2(\alpha_2+\gamma_2)(\beta_2+\delta_2) + (\alpha_2+\beta_2+\gamma_2+\delta_2)\epsilon_2} N$$

$$+ \frac{2\gamma_2(\beta_2+\delta_2) + (\gamma_2+\delta_2)\epsilon_2}{2(\alpha_2+\gamma_2)(\beta_2+\delta_2) + (\alpha_2+\beta_2+\gamma_2+\delta_2)\epsilon_2} \alpha_1$$

$$+ \frac{2\delta_2(\alpha_2+\gamma_2) + (\gamma_2+\delta_2)\epsilon_2}{2(\alpha_2+\gamma_2)(\beta_2+\delta_2) + (\alpha_2+\beta_2+\gamma_2+\delta_2)\epsilon_2} \beta_1$$

$$+ \frac{2\alpha_2(\beta_2+\delta_2) + (\alpha_2+\beta_2)\epsilon_2}{2(\alpha_2+\gamma_2)(\beta_2+\delta_2) + (\alpha_2+\beta_2+\gamma_2+\delta_2)\epsilon_2} \gamma_1$$

$$+ \frac{2\beta_2(\alpha_2+\gamma_2) + (\alpha_2+\beta_2)\epsilon_2}{2(\alpha_2+\gamma_2)(\beta_2+\delta_2) + (\alpha_2+\beta_2+\gamma_2+\delta_2)\epsilon_2} \delta_1$$

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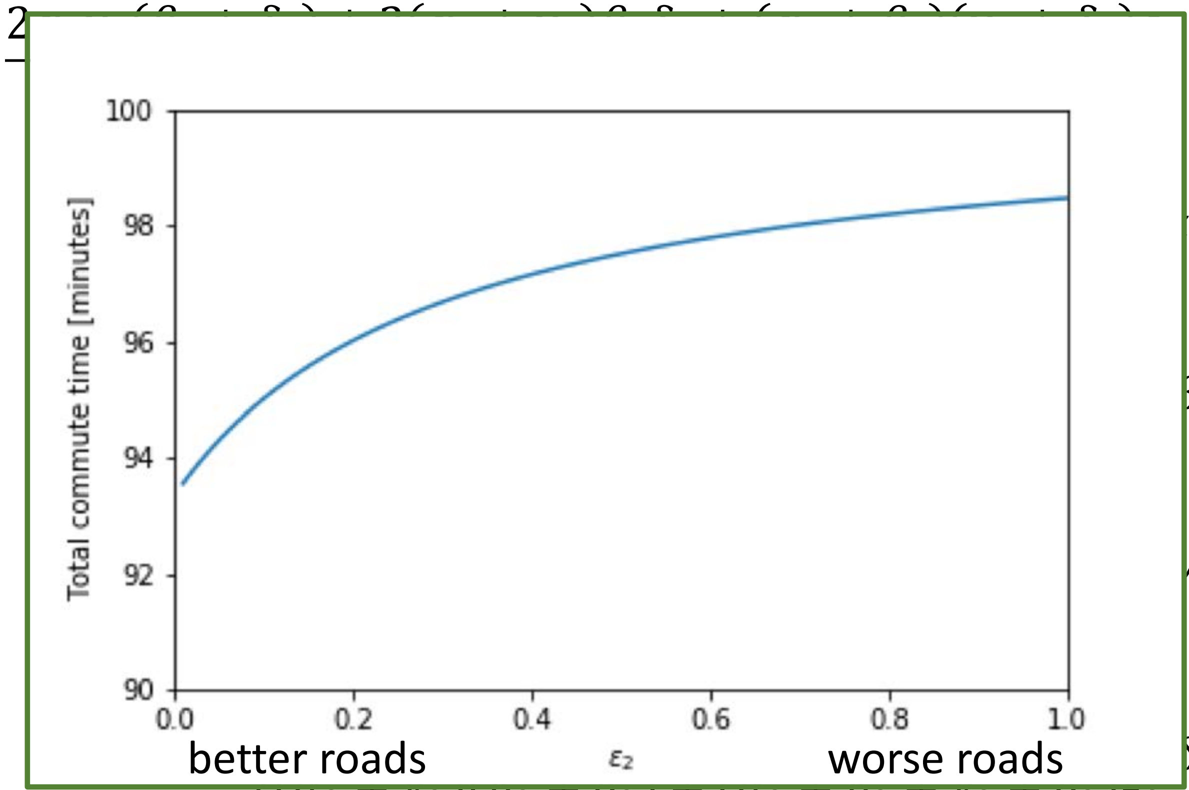
$T =$



N

$2(\alpha_2 + \gamma_2)(\beta_2 + \delta_2) + (\alpha_2 + \beta_2 + \gamma_2 + \delta_2)\epsilon_2$

Long roads with congestion

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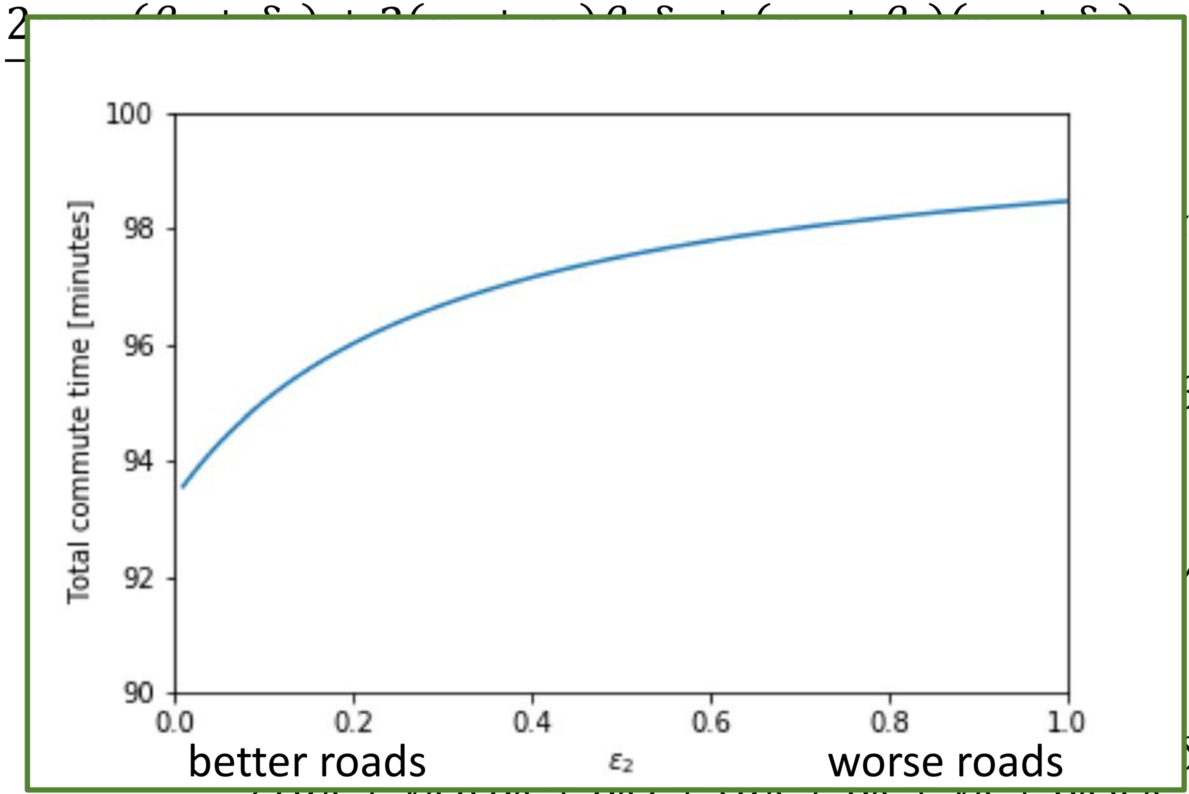

Increasing congestion on the connector (larger ϵ_2) means longer commute times.

No paradox here.

So where does this leave Braess' paradox?

Braess said that removing roads can reduce commute times.

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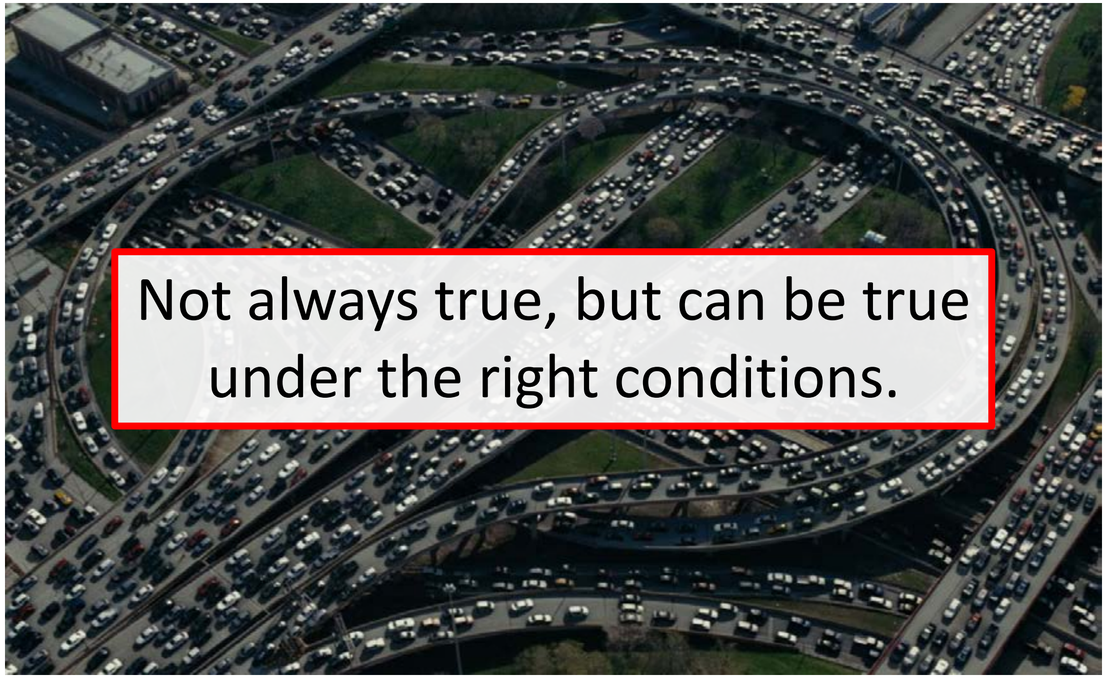


Braess said that removing roads can reduce commute times.

Urban experiments demonstrating the paradox

- **Stuttgart, Germany**

In 1969 a section of new freeway was closed and traffic immediately improved. This road was closed permanently after this effect was observed.

Knödel W (1969). Graphentheoretische Methoden und ihre Anwendungen. Pages 57–59. Springer-Verlag.

- **Seoul, South Korea**

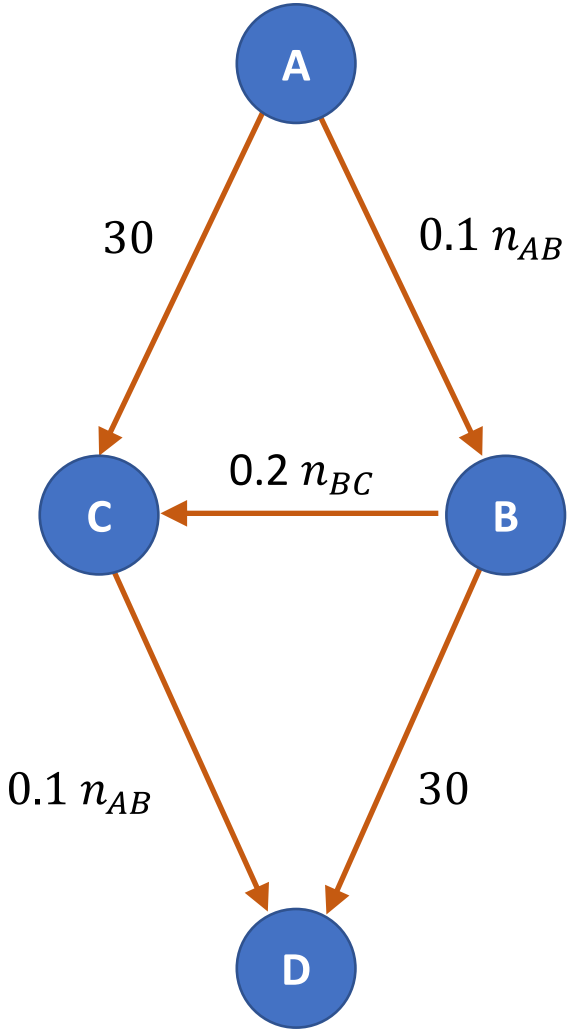
The 2003 Cheonggyecheon Restoration Project removed an elevated highway to restore a waterway that had been built-over in earlier decades. A result of the removal of this road was a decrease in commute times.

Easley, D and Kleinberg, J: "Networks", page 71. Cornell Store Press, 2008

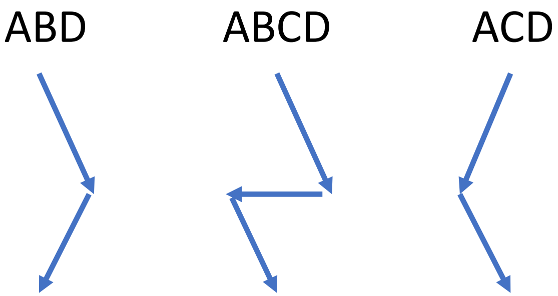
- **New York City**

Congestion significantly improved in central Manhattan when the traffic patterns on 42nd Street were modified in 1990 and Times Square was closed in 2008.

Why does the paradox occur?



1. Only three paths are actually used:



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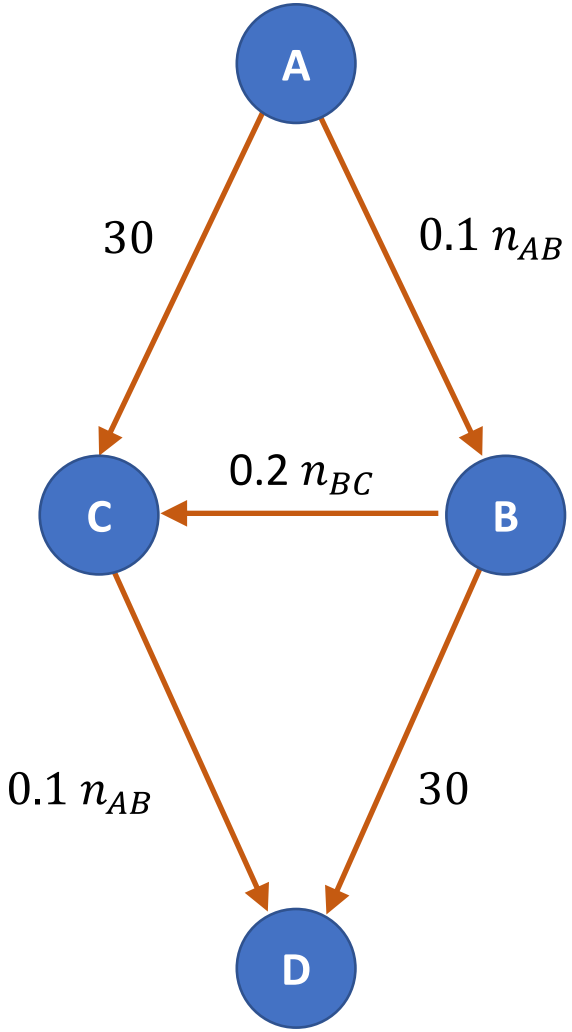

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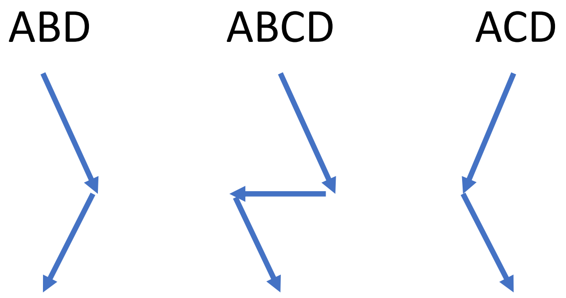
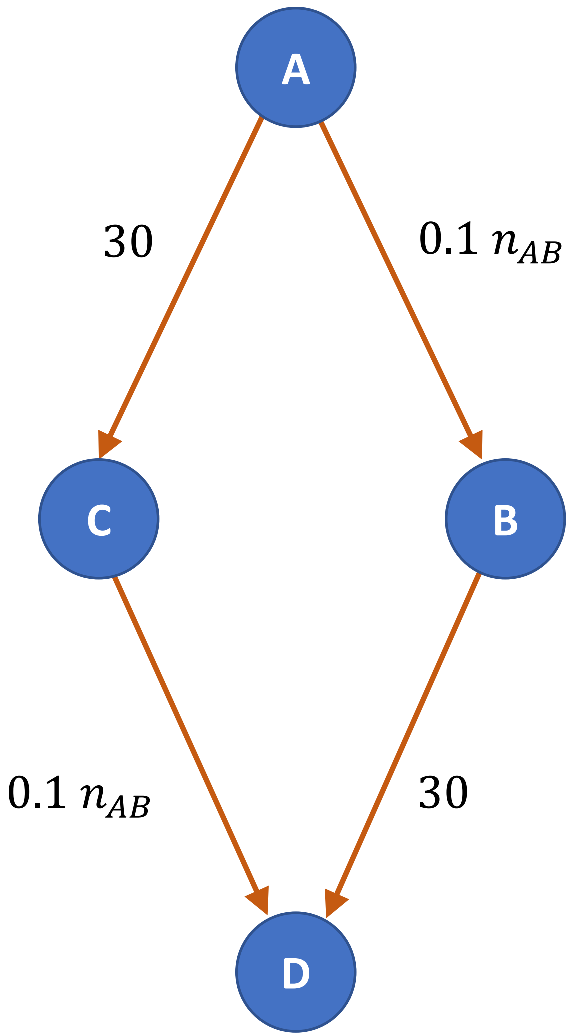


2. We now consider what happens if we at first do not have a connector road, and then a second case where the connector is added.

Why does the paradox occur?



3. If the connector road is bad or nonexistent, then commuters use only ABD and ACD.

For N = 200 commuters, 100 will take each path:

$$T = 30 + 0.1 \times 100 = 40 \text{ minutes}$$

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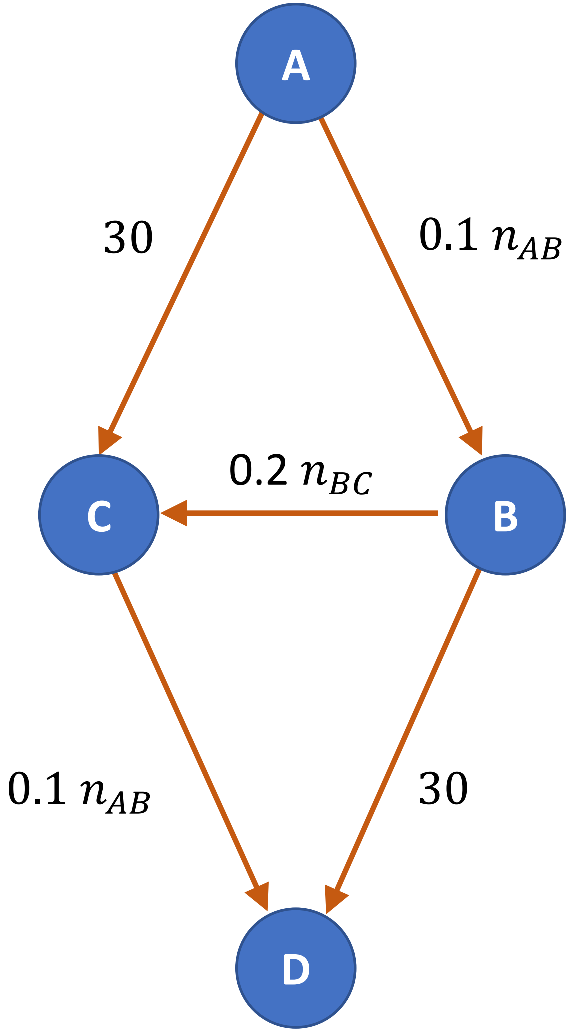

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For N = 200 commuters, 100 will take each path:

$T = 30 + 0.1 \times 100 = 40$ minutes

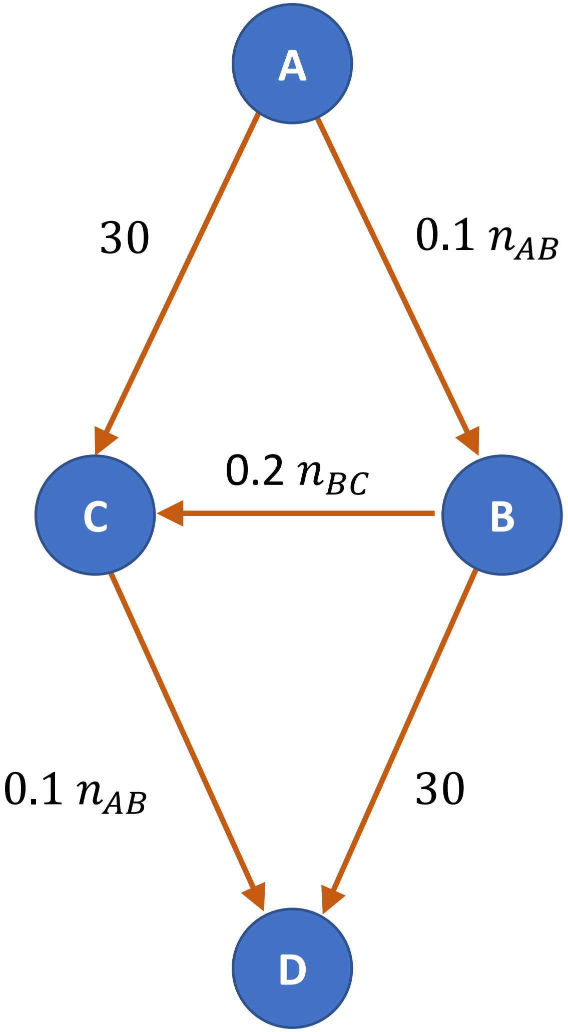
4. If the connector road is present (say ϵ_2=0.2) then people cram onto the connector road to get to work as quickly as possible.

$n_{ABD} = 60$ $\quad T = 0.1(60+80) + 30 = 44$ minutes

$n_{ABCD} = 80$ $\quad T = 0.1(60+80) + 0.2(80) + 0.1(60+80) = 44$ minutes

$n_{ACD} = 60$ $\quad T = 30 + 0.1(60+80) = 44$ minutes

Why does the paradox occur?



3. If the c … xistent, then c … ACD.

For N = 2 …

$T = 30$ …

4. If the c … ϵ_2=0.2) then p … r road to get to …

$n_{ABD}\ =$ … es

$n_{ABCD} =$ … $60 + 80) = 44$ minutes

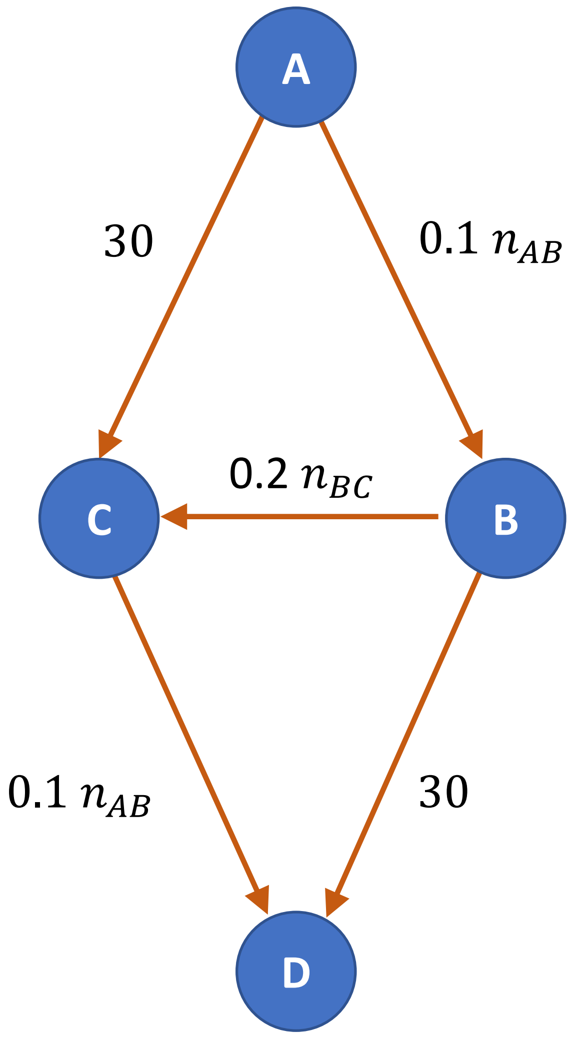
$n_{ACD}\ =$ … es

The total commute time increases by 4 minutes with a connector road.

This happens because each person makes the very logical decision to minimize his/her own commute time.

People will take those roads that offer shorter absolute travel times, resulting in greater overall congestion.

Remember the prisoner's dilemma:



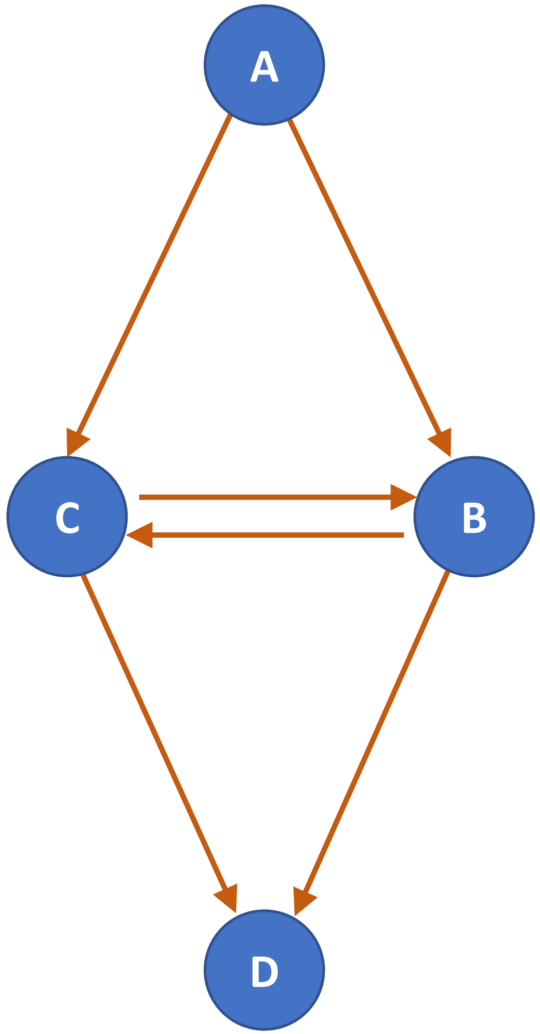
Prisoner B choices

Prisoner A choices		loyal	defect
	loyal	**2**	**10**
	defect	**0**	**4**

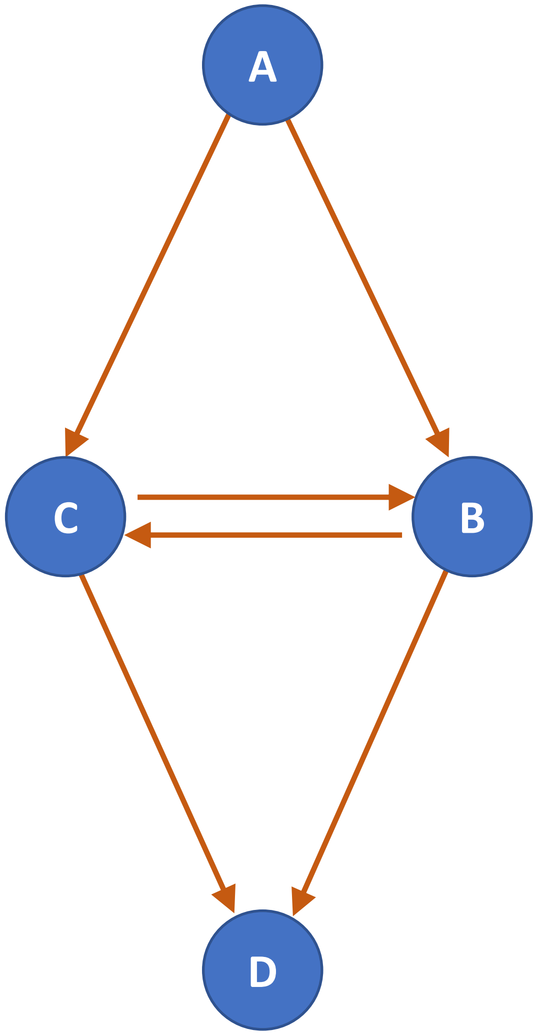
Nash equilibrium point

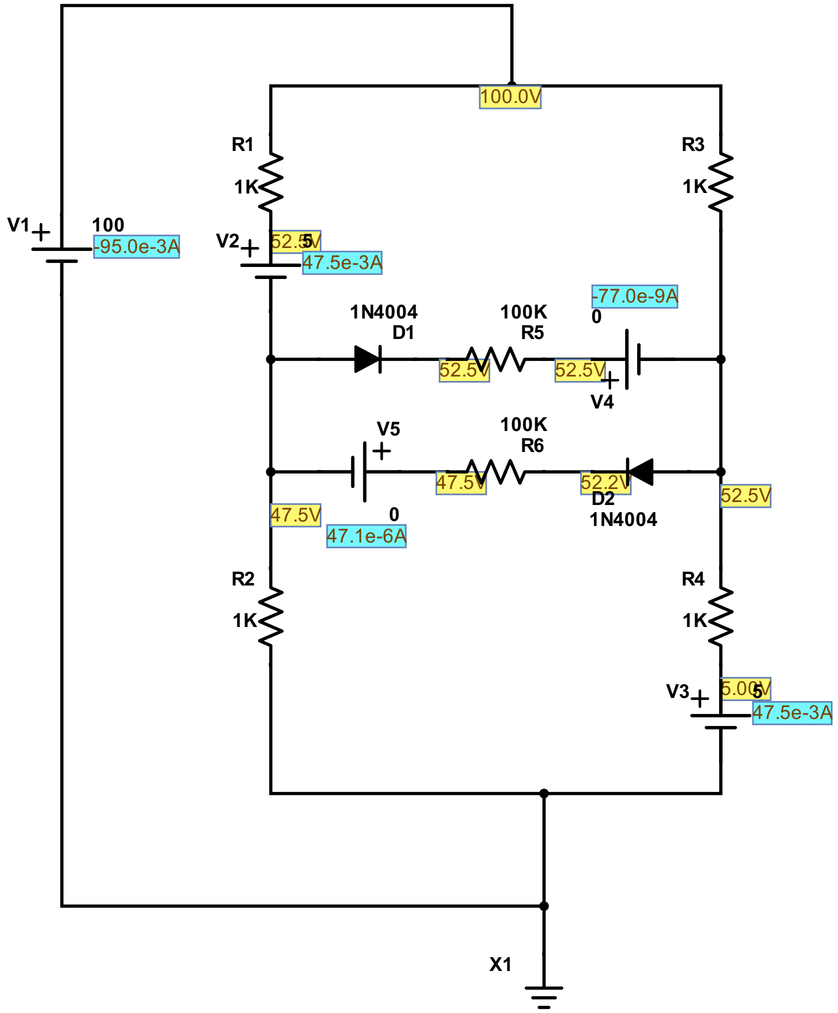
Creating a city in the lab

Creating a city in the lab

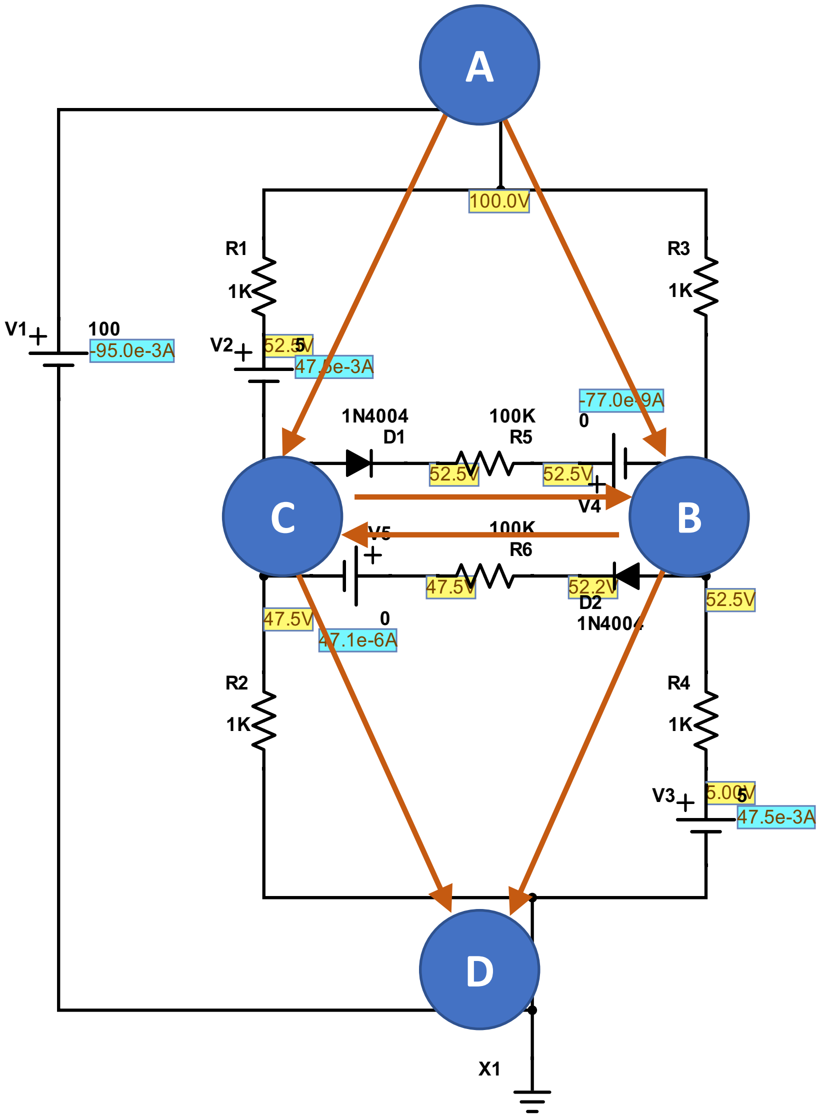


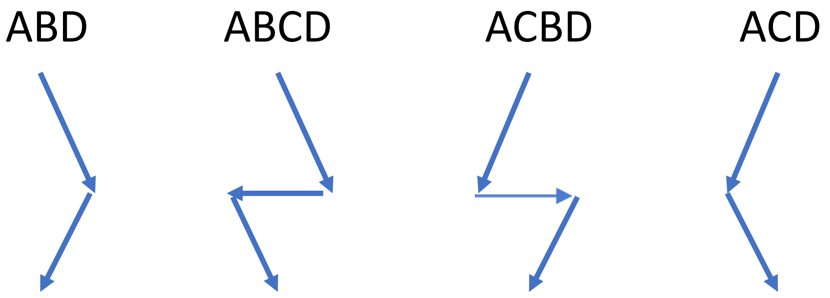
Creating a city in the lab





Creating a city in the lab





$$I_{tot} = I_{ABD} + I_{ABCD} + I_{ACD} + I_{ACBD}$$

ABD: $V_0 = V_1 + R_1(I_{ABD} + I_{ABCD}) + V_2 + R_2(I_{ABD} + I_{ACBD})$

ABCD: $V_0 = V_1 + R_1(I_{ABD} + I_{ABCD}) + V_5 + R_5 I_{ABCD} + V_4 + R_4(I_{ACBD} + I_{ACD})$

ACBD: $V_0 = V_3 + R_3(I_{ACD} + I_{ACBD}) + V_6 + R_6 I_{ACBD} + V_2 + R_2(I_{ACD} + I_{ABCD})$

ACD: $V_0 = V_3 + R_3(I_{ACD} + I_{ACBD}) + V_4 + R_4(I_{ACD} + I_{ABCD})$

Creating a city in the lab

$$\begin{pmatrix} N \\ \alpha_1+\beta_1 \\ \alpha_1+\delta_1+\epsilon_1 \\ \beta_1+\gamma_1+\theta_1 \\ \gamma_1+\delta_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & -(\alpha_2+\beta_2) & -\alpha_2 & -\beta_2 & 0 \\ 1 & -\alpha_2 & -(\alpha_2+\delta_2+\epsilon_2) & 0 & -\delta_2 \\ 1 & -\beta_2 & 0 & -(\beta_2+\gamma_2+\theta_2) & -\gamma_2 \\ 1 & 0 & -\delta_2 & -\gamma_2 & -(\gamma_2+\delta_2) \end{pmatrix} \begin{pmatrix} T \\ n_{ABD} \\ n_{ABCD} \\ n_{ACBD} \\ n_{ACD} \end{pmatrix}$$

$$\begin{pmatrix} I_{tot} \\ V_1+V_2 \\ V_1+V_4+V_5 \\ V_2+V_3+V_6 \\ V_3+V_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & -(R_1+R_2) & -R_1 & -R_2 & 0 \\ 1 & -R_1 & -(R_1+R_4+R_5) & 0 & -R_4 \\ 1 & -R_2 & 0 & -(R_2+R_3+R_6) & -R_2 \\ 1 & 0 & -R_4 & -R_3 & -(R_3+R_4) \end{pmatrix} \begin{pmatrix} V_0 \\ I_{ABD} \\ I_{ABCD} \\ I_{ACBD} \\ I_{ACD} \end{pmatrix}$$

Concluding remarks

- Braess' paradox exists under specific conditions, which have been observed in actual cities.
- Understanding why the paradox arises in this simple model works provides some insights into human behavior and the trappings of unregulated competition.
- Agents that act independently and without communication often make collective decisions that are sub-optimal (i.e. the prisoners).
  - In this way, commuters exhibit collective behavior that is comparable to that of electrons that as dictated by external forces and absent a sense of will to change their fate.
- Many ways to build upon this model:
  - Different origins and/or destinations, fluctuations in decisions, the psychology of particular agents and willingness to experiment, the process by which a system comes into equilibrium, etc.

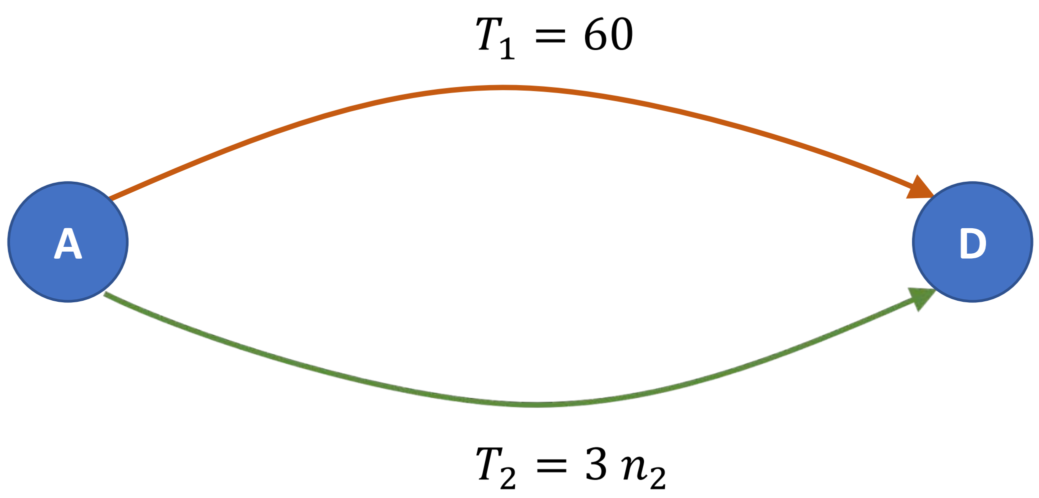
Fluctuation level assuming random choices

With N=100, we found:

$T = 60$ minutes

$n_1 = 80$

$n_2 = 20$



We can say that each commuter ascribes probabilities to each path:

$p_1 = 0.80$

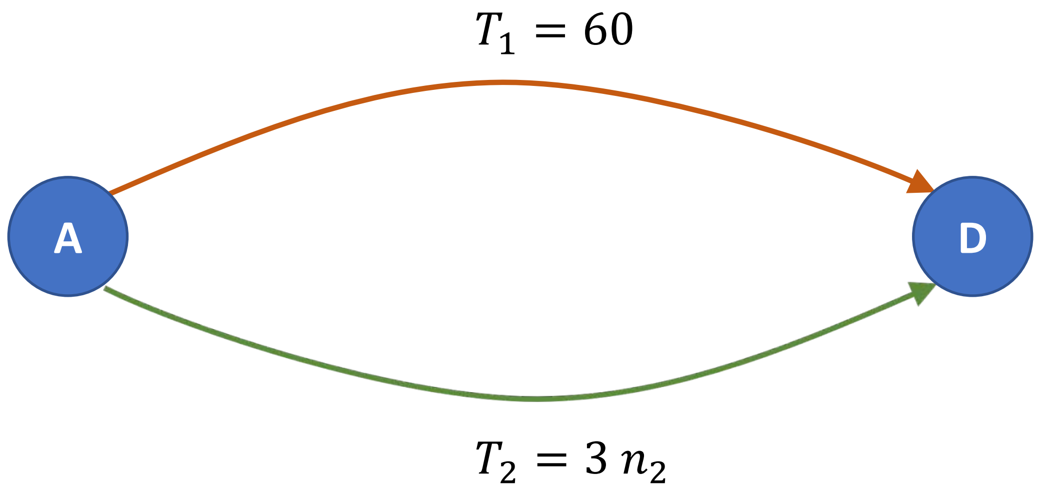
$p_2 = 0.20$

We can now investigate the fluctuations in the system, because it is not true that a bunch of people making independent decisions always has this same division between paths.

Fluctuation level assuming random choices

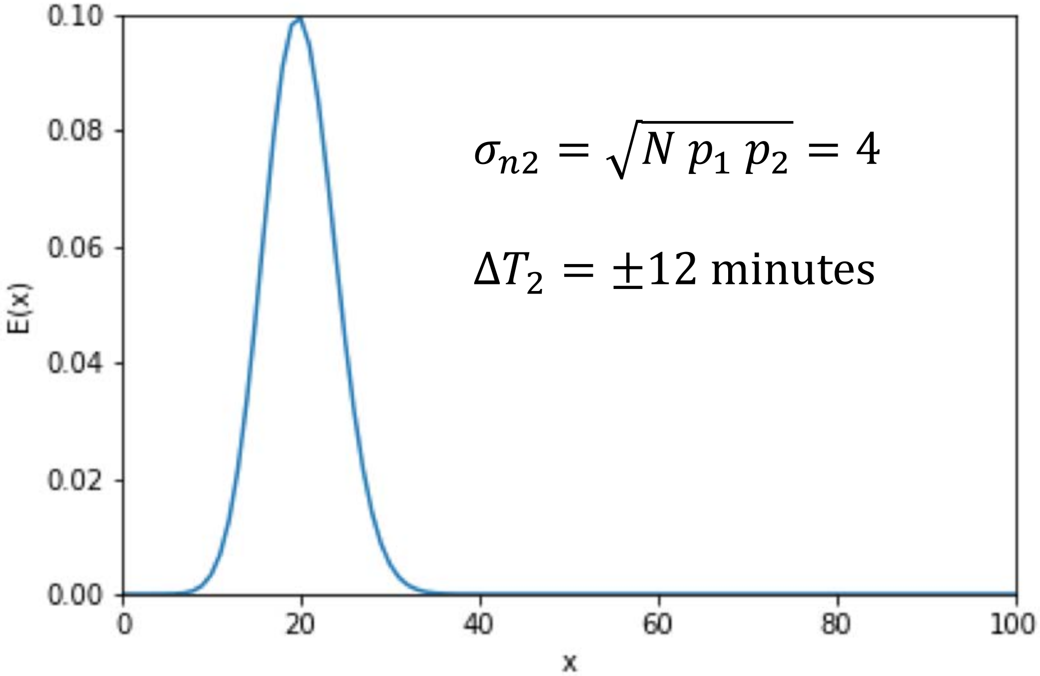
What is the expected fluctuation in the commute times?

This is equivalent to asking what is the probability of that x commuters choose path 2 if they independently make such a decision based on these probabilities?



Binomial distribution:

$$E(x) = \frac{N!}{x!\,(N-x)!}\, p_1^{N-x}\, p_2^{x}$$



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Let us hypothesize an inverse linear relationship:

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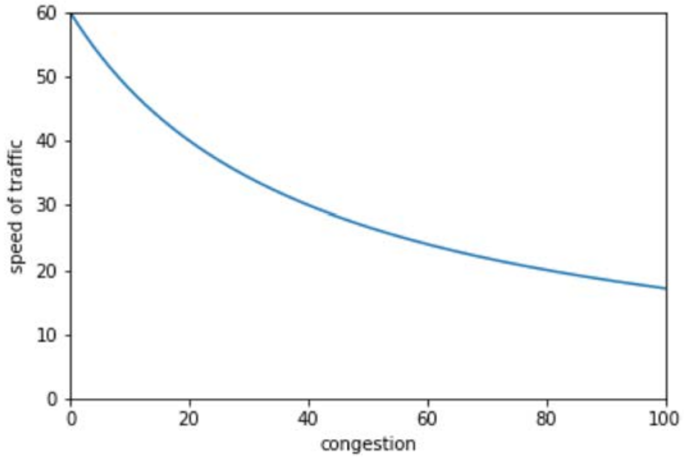

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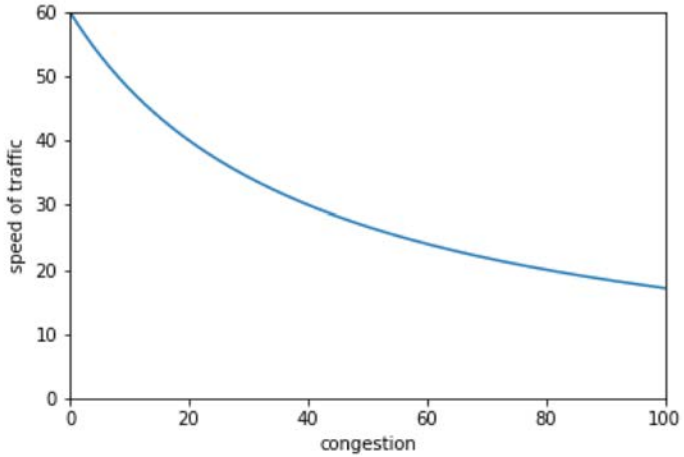

The travel time is:

$$T_1 = \frac{L_1}{v_0}(1 + a\, n_1)$$

$$= \alpha_1 + \alpha_2\, n_1$$

How do we make sense of this problem?

- Solve for the conditions under which Braess's paradox holds:
  - We can find the conditions under which the commute time decreases with increasing congestion on the connector.

Set: $\frac{dT}{d\epsilon_2} < 0$ ← Solve for the general conditions in the 10-dimensional parameter space

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Let's try it.

10-dimensional algebra…

- We would like to know under what conditions (what values of the α_1, α_2, β_1, etc.) does our system exhibit this paradox.

- Let ϵ_2 be our control parameter that accounts for congestion:

  good roads = small ϵ_2 & bad roads = large ϵ_2

- What we really want to know is whether an increase in ϵ_2 leads to a decrease in commute times.
  - Need to calculate, and then evaluate, the function: $\frac{dT}{d\epsilon_2} < 0$

10-dimensional algebra… and two cases.

- Case 1: asymmetric connectors

$$\frac{dT}{d\epsilon_2} = \frac{BC - AD}{(C + D\epsilon_2)^2}$$

- With Classic Braess values ($\alpha_1 = \beta_2 = \gamma_2 = \delta_1 = \epsilon_1 = 0$):

$$BC - AD = \theta_2(\alpha_2\delta_2 - \beta_2\gamma_2)\begin{bmatrix} (\alpha_2\delta_2 - \beta_2\gamma_2)N + (\beta_2 + \delta_2)(\alpha_1 + \beta_1) \\ +(\alpha_2 + \gamma_2)(\gamma_1 + \delta_1) - (\alpha_2 + \beta_2 + \gamma_2 + \delta_2)(\alpha_1 + \delta_1 + \epsilon_1) \end{bmatrix}$$

$$BC - AD = \theta_2(\alpha_2\delta_2)[\alpha_2\delta_2 N + \beta_1\delta_2 + \alpha_2\gamma_1] > 0$$

10-dimensional algebra… and two cases.

- Case 2: symmetric connectors (θ=ϵ)

$$\frac{dT}{d\epsilon_2} = \frac{BC - AD}{(C + D\epsilon_2)^2}$$

- With Classic Braess values (α_1 = β_2 = γ_2 = δ_1 = ϵ_1 = 0):

$$BC - AD = (\alpha_2\delta_2 - \beta_2\gamma_2)\begin{bmatrix} 2(\alpha_2\delta_2 - \beta_2\gamma_2)N + 2(\beta_2 + \delta_2)(\alpha_1 + \beta_1) \\ +2(\alpha_2 + \gamma_2)(\gamma_1 + \delta_1) + (\alpha_2 + \beta_2 + \gamma_2 + \delta_2)(\beta_1 + \gamma_1 - \alpha_1 - \delta_1) \end{bmatrix}$$

$$BC - AD = (\alpha_2\delta_2)[2\alpha_2\delta_2 N + 2\beta_1\delta_2 + 2\alpha_2\gamma_1 + (\alpha_2 + \delta_2)(\beta_1 + \gamma_1)] > 0$$

How do we resolve?

Notice that commute time does not depend on ϵ_1.

- We want to find the case where the time decreases with increasing congestion.
- Set $\frac{dT}{d\epsilon_2} < 0$
- This gives us the condition that we should also have

$$T < \frac{[2(\alpha_2 + \gamma_2)(\beta_2 + \delta_2) + (\alpha_2 + \beta_2 + \gamma_2 + \delta_2)\epsilon_2][(\alpha_1 + \beta_1)(\gamma_2 + \delta_2) + (\alpha_2 + \beta_2)(\gamma_1 + \delta_1)(\alpha_2 + \beta_2)(\gamma_2 + \delta_2)]}{\alpha_2 + \beta_2 + \gamma_2 + \delta_2}$$

Commuting from A to D with intermediate points

- Welcome to the 5th dimension!

The ``Classic Braess" scenario: