A $b-1$ multiplicative identity for digit summing in base b

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1 Introduction

Many years ago a friend made a curious statement to me regarding the preferred base for counting. He pointed out that 9 multiplied by any integer, except 0, yields an integer whose sum of digits equals 9. For example, $9 \times 53 = 477$, the sum of the digits being $4 + 7 + 7 = 18$, and when the summing is applied recursively gives $1 + 8 = 9$. My friend's claim was that this indicates some kind of cosmic preference for base 9 and concluded that we are mistaken in our adherence to the base 10 counting system. The proof presented here shows that this property of summing digits after multiplication by 9 is indeed true for base 10, and more generally, that the property holds for multiplication by $b - 1$ when counting in base b.

2 Statement of the theorem

Let $N \in \mathbb{N}^+$ be a natural number under investigation, $b \in \mathbb{N}^+ - \{1\}$ is the base in which N will be represented, and let $\mathbb{B} = \{0, \ldots, b-1\}$ be the set of integers used in this base b representation. The representation of N in base b is specified by a sequence of digits, with each digit an element of $\mathbb{B}$. Equivalently, N can be represented in summation form as

$$N = \sum_{i=0}^{l} n_i b^i \tag{1}$$

where the coefficients $n_i \in \mathbb{B}$ are the numerical coefficients of the digit representation of N in base b and $l \in \mathbb{N}^0$ is the index for which for all $i > l$ we have $n_i = 0$. We define a mapping function $S(N; b) : \mathbb{N}^+ \to \mathbb{N}^+$, the process described in the introduction, as follows

$$S(N; b) = \sum_{i=0}^{l} n_i \tag{2}$$

where $S(N; b)$ maps the base b representation of N to a single number. Let $\mathbb{Q}\,(N; b)$ be the sequence of recursive application of $S(N; b)$, that is, $\mathbb{Q}\,(N; b) = \{S(N; b), S(S(N; b)), \ldots\}$. The $b-1$ multiplication theorem is that the limit of $\mathbb{Q}(M; b)$ converges to $b-1$ for all for $M = N(b-1)$ for any base $b \geq 2$.

3 Proof

We define the function $F(N;b) = N - S(N;b)$, which can be expressed as

$$F(N;b) = \sum_{i=0}^{l} n_i \left(b^i - 1\right) \tag{3}$$

and is clearly positive for all N that have more than one digit in their base b representation, equivalent to the condition $l \geq 1$. Therefore, $S(N;b) < N$ for all $N \geq b$ and it follows that $\mathbb{Q}(N;b)$ decreases monotonically for $N \geq b$. Since $F(N;b) = 0$ for $N < b$, we have that the limit of $\mathbb{Q} \in \mathbb{B}^+ = \{1, \ldots, b-1\}$ for all N since at least one of the n_i must be non-zero.

In the following we consider $\mathbb{Q}(M;b)$ for $M = N(b-1)$, and define the base b representation of M similar to that for N used previously, only replacing the n_i of Eq. 1 with m_i.

We first consider the proof for $b = 2$ as a special case. For $b = 2$, we have $b - 1$ equal to the multiplicative identity element. The product we seek, $M = N(b-1)$, is therefore equal to N. Recalling the prior result that the limit of $\mathbb{Q}(N;b) \in \mathbb{B}^+$, for base 2 this implies that the limit of $\mathbb{Q}(N,2) \in \{1\}$, and therefore, the limit of $\mathbb{Q}(N;b)$ for all N is equal to 1, and the theorem is proved for $b = 2$.

To proceed with the proof for $b \geq 3$, we divide $\mathbb{N}^+$ into subsets based on the length of the digit representation in base b. That is, we define the set of j-digit representations in base b as $\mathbb{X}_j = \{b^{j-1}, \ldots, b^j - 1\}$ for $j \in \mathbb{N}^+$, where we note that $\mathbb{N}^+ = \sum_j^\infty \mathbb{X}_j$. We proceed by considering $N \in \mathbb{X}_1$. Note that if $N = 1$, then $M = b - 1$ and $S(M;b) = b - 1$. For $N \geq 2$ we need the base b digit representation of M to calculate $S(M;b)$. To derive this form we rewrite M as follows

$$\begin{aligned} M &= n_0 \left(b - 1\right) & (4) \\ &= \left(n_0 - 1\right) b^1 + \left(b - n_0\right) b^0 & (5) \end{aligned}$$

where the second form is allowed because we know that $n_0 \neq 0$ since $N \in \mathbb{X}_1$. Thus, the coefficient of the b^1 term in Eq. 5 can be identified as m_1 and the b^0 coefficient as m_0 since they are both positive given that $n_0 \in \mathbb{B}^+$ since $n_0 \geq 1$. It now follows that $S(M;b) = (n_0 - 1) + (b - n_0) = b - 1$, and the theorem is proved for all $N \in \mathbb{X}_1$. We now extend the results to $N \in \mathbb{X}_2$. As before, we rewrite M as

$$\begin{aligned} M &= \left(n_1 b + n_0\right)\left(b - 1\right) & (6) \\ &= n_1 b^2 + \left(n_0 - n_1\right) b^1 - n_0 b^0 & (7) \end{aligned}$$

and now have to assess whether all of the coefficients of the b^i in Eq. 7 are positive. If $n_0 = 0$ then we need not modify the b^0 coefficient. This, however, requires that the second coefficient is necessarily negative, and so we must add b to it to make it positive, and in turn subtracting 1 from the b^2 coefficient, to yield

$$M = (n_1 - 1)\, b^2 + (b + n_0 - n_1)\, b^1 - n_0 b^0 \tag{8}$$

where now all coefficients are positive and can be associated with the m_i. It follows that $S(M;b) = b-1$ for this case. If, on the other hand, $n_0 \neq 0$ then we must add b to the b^0 coefficient and subtract 1 from the b^1 coefficient to get

$$M = n_1 b^2 + (n_0 - n_1 - 1)\, b^1 + (b - n_0)\, b^0 \tag{9}$$

and now need to asses whether the b^1 coefficient is negative. If $n_0 \geq n_1 + 1$, then each coefficient is an element of $\mathbb{B}$, and these coefficients then represent the m_i coefficients of the base b representation of M. Therefore, $S(M;b) = n_1 + (n_0 - n_1 - 1) + (b - n_0) = b - 1$. If, on the other hand, $n_0 < n_1 + 1$ then the b^1 coefficient may be made positive by adding b to it, since $(n_1 + 1) \in \{1, \ldots, b\}$, to get the following form

$$M = (n_1 - 1)\, b^2 + (b + n_0 - n_1 - 1)\, b^1 + (b - n_0)\, b^0 \tag{10}$$

where each coefficient must now be an element of $\mathbb{B}$ and can therefore be identified with the m_i. In this case we have $S(M;b) = (n_1 - 1) + (b + n_0 - n_1 - 1) + (b - 1) = 2(b-1)$. Given that we are considering $b \geq 3$, and by the results of $N \in \mathbb{X}_1$, we know that $S(2(b-1);b) = b-1$, the theorem is proved for all $N \in \mathbb{X}_2$ and for all b.

To extend the theorem to all $\mathbb{X}_j$, we must generalize the methods used for analysis of $N \in \mathbb{X}_1$ and $N \in \mathbb{X}_2$. The general polynomial form of $M = N(b-1)$ for an $N \in \mathbb{X}_j$ with $j \geq 3$ is

$$\begin{aligned} M &= (n_j b^j + n_{j-1} b^{j-1} + \ldots + n_0 b^0)(b-1) & (11) \\ &= n_j b^{j+1} + (n_{j-1} - n_j)\, b^j + \ldots - n_0 b^0 & (12) \end{aligned}$$

where we have $j+1$ terms in this polynomial representation and we have yet to ensure that the coefficients of the b^i of Eq. 12 are elements of $\mathbb{B}$. We can systematically transform the coefficients of the b^i to make sure that each of the coefficients is an element of $\mathbb{B}$ by progressing from b^0 to b^{j+1}.

In the form of Eq. 12, the sum of the coefficients of the b^i is equal to $n_j + (n_{j-1} - n_j) + \ldots - n_0 = 0$. As before, if $n_0 \neq 0$, we may transform Eq. 12 by adding b to the b^0 coefficient and subtracting 1 from the b^1 coefficient. The sum of the coefficients in this representation of M is now $b-1$. Upon inspecting the b^1 coefficient, should it not be an element of $\mathbb{B}$, then we add a b to the b^1 coefficient and subtract a 1 from the b^2 coefficient. The sum of the coefficients would now be equal to $2(b-1)$. In this this way we may proceed through the first j terms until all coefficients are elements of $\mathbb{B}$. If, on the other hand, $n_0 = 0$ then we can leave it unmodified and proceed to the next term. The b^1 coefficient either has $n_1 = 0$, for which we can proceed to the next term in the sequence, or else it is non-zero and we must make it positive by adding b to it and subtracting 1 from the coefficient of the b^2 coefficient. We must have at least one non-zero n_i, and therefore there must be at least one occurrence for which the coefficients must be modified as outlined here. Therefore, it follows that $S(M;b) = k(b-1)$ where $k \in [1, \ldots, j]$ for $M = N(b-1)$ and all $N \in \mathbb{X}_j$.

We have proved that the limit of $\mathbb{Q}(N;b) = b-1$ for $N \in \mathbb{X}_1$ and $N \in \mathbb{X}_2$, and that $S(M;b) = k(b-1)$ where $k \in [1, \ldots, j]$ for $N \in \mathbb{X}_j$. We first note that $3 \in \mathbb{X}_1$ or $3 \in \mathbb{X}_2$ for all $b \geq 3$.

Therefore, because $S(M;b)$ is at most $3(b-1)$ for $M = N(b-1)$ and $N \in \mathbb{X}_3$, it follows that the limit of $\mathbb{Q}(N;b) = b-1$ for $N \in \mathbb{X}_j$ with $j \in \{1,2,3\}$. By continuation, since $j \in \sum_1^j \mathbb{X}_j$ for all $j \geq 3$, it follows that the limit of $\mathbb{Q}(N;b)$ is $b-1$ for all $N \in \mathbb{N}^+$ and all $b \geq 2$, and the theorem is proved.

4 Conclusions

We have proven that recursive digit summing of any $b-1$ multiple of a number N results in a value of $b-1$ when N is represented in base b. Therefore, the generalization of my friend's assertion, that we should count in base 9, is that we should count in base $b-1$. Applied recursively, the limit of this logic is that the preferable base for counting is base 1.