MEASUREMENT UNCERTAINTIES

What distinguishes science from philosophy is that it is grounded in experimental observations. These observations are most striking when they take the form of a quantitative measurement. These measurements have the power to dismiss a faulty theory or lend support to a theory with which it is consistent. Measurements, however, are never exact and to know how much faith to put into the measurement we must know "how good" the experimental measurement is in shedding light on the theory we wish to test. This idea of goodness is really an amalgam of multiple issues.

Generally speaking there are three broad issues at play, accuracy of your measurement, the precision of your measurement, and the applicability of the theory you wish to test in the situation at hand. Flaws related to accuracy tend to fall in the category of systematic error, which is notoriously difficult to identify. The challenge in identifying systematic error is that, in many cases, you must essentially know what you don't know. Much easier to identify are limits to precision which pop out at you as either limits in your measuring device (akin to a blurriness in your vision) or as variability in the quantity you are trying to measure. Systematic error doesn't pop out at you because the error tends to be repeatable. When it comes to the applicability of your theory, this is a question of how well we expect our model to represent reality. Almost always there are simplifying assumptions we must make to apply our theory to the situation at hand. One needs to think carefully about how reasonable those assumptions are. We as scientists must have an honest means of describing these limitations.

For example, imagine we want to measure the length of a piece of paper with a ruler. In order to make that measurement, we have to accurately line up the zero mark at one end of the paper. We have to have the ruler parallel to the edge of the paper. We have to be able to estimate where the other end of the paper hits the ruler. We can't do any of those things perfectly.

On the ruler in our minds the smallest division on a ruler is a millimeter. It would be dishonest to imagine that we could measure something with this ruler any more precisely than a few tenths of a millimeter. As a scientist you must make an honest judgment about the accuracy of this measurement process, in light of this a good estimate of our uncertainty in measuring with a ruler is (under the best conditions) 0.3 mm. If our best estimate for the length of this imaginary piece of paper is 77.1mm we would then report the length as

$$L = 77.1 \pm 0.3 \text{ mm}.$$

The absolute uncertainty in this measurement is ± 0.3 mm. In stating that, we are claiming that due to the imperfectness of the measuring process, the actual length could be anywhere between 76.8 mm and 77.4 mm. Note that it is an estimate of the precision with which we made our measurements due to the resolution of our measuring device and is a source of uncertainty we will deal with quantitatively. We will deal with estimating other sources of uncertainty later.

More on Reporting Uncertainty—Significant Figures

Even without explicitly listing the uncertainty, you describe something about the preciseness of measurements by the way you write the numbers. A length of "77.1 mm" implies that the length is closer to 77.1 mm than it is to 77.0 mm or 77.2 mm. Likewise a length of "77.10 mm" implies that the length is closer to 77.10 mm than it is to 77.09 mm or 77.11 mm, whereas "77 mm" implies that the length is closer to 77 mm than it is to 76 m or 78 mm. The precision of the measurement is conveyed by the number of digits used in writing the number. The digits used to indicate this precision are called significant digits or significant figures. So a length of "77.10 mm" has four significant figures, while the length of "77 m" has only two.

Not all the digits written in a number are necessarily "significant." Those zeroes that only locate the decimal point are not significant. A measurement reported as "0.00020 m" has only two significant digits. The three zeroes after the decimal point are not significant. However, a measurement of "1.00020 m" would have six significant digits. Thinking about it in terms of scientific notation clarifies the difference between these two cases. The one measurement would be "2.0×10^{-4} m", while the second would still be "1.00020 m".

Similarly, there is an ambiguity about a measurement like "2000 m". It is not clear how many of those digits are significant! Writing large numbers in scientific notation allows us to determine how many of those zeroes are significant. If the measurement is written as "2.0×10^{3} m", then we can see that there are only two significant digits.

Do not confuse significant figures and precision with the largeness or smallness of a number. The number 0.02 s is small, but not precise, since it only has one significant figure. If the experiment warrants it, you should be sure to keep the appropriate significant figures, so the number might be 0.0253 s.

<u>Comparisons</u>

You will frequently need to compare your measured values (or a number calculated from your measured value) to some other value. Examples of this would be to compare the measured value to (a) a theoretical value (b) a value obtained by a different measurement technique or (c) some "accepted" value.

In real experimental measurements, the comparisons (a) and (b) are the most frequent. In the elementary lab, you do encounter a number of situations in which there is some reference or accepted value. This is really just a comparison with some other measurement that has been done much more precisely than yours. If we measure the acceleration due to gravity in the lab, there is an accepted value to compare to because it has been measured many times with great precision.

Your experimental result will not match **perfectly** with whatever you're comparing it to. The question we're always interested in is <u>do they agree within the uncertainty of the measurements</u>?

For example, if we measure the acceleration due to gravity as $9.74\pm0.09\ \mathrm{m/s^2}$, we didn't get the same exact number as the accepted value of $9.81\ \mathrm{m/s^2}$. However, given our stated uncertainty, our measurement has yielded a range of acceptable values, the actual result of our measurement could be anywhere between $9.65\ \mathrm{m/s^2}$ and $9.83\ \mathrm{m/s^2}$. Since our range overlaps with the very narrow range of accepted value, our measurement is consistent with the expected result.

<u>Estimating Measurement Uncertainty</u>

How do you determine the uncertainty of your measurement? On what factors is it based? The uncertainty is a combination of two things: the limiting precision of your measuring instrument and the limitations of the experiment (including the experimenter).

Let us return to our initial example of measuring a piece of paper, we have already seen how the limited precision of a ruler produces some uncertainty in any measurement. A ruler might have a precision no better than a few tenths of a millimeter, a scale might read masses only to the nearest tenth of a gram, or a timing mechanism might be accurate to a millisecond. Each of these instruments sets a fundamental limit on how reliably a quantity can be measured. A ruler gives a small uncertainty when measuring the length of a piece of paper, but for the thickness of a piece of paper, a ruler will not do. An instrument with much finer resolution is required—like a micrometer.

In most real cases, your uncertainty may be greater than that due to the instrument. Imagine using a meter stick to measure the length of several pieces of paper. The lengths in our imaginary stack are not the same to a few tenths of a millimeter. If we repeat the measurement in the freezer the ruler will have shrunk and change the results. All of this must be considered when getting a realistic estimate of the uncertainty. So how do you make this realistic estimate?

(1) Do consider the fundamental limitations of the measuring device.

(2) Beyond that, observe the difficulties you have in making the measurement. You are part of the measuring technique. If there is some difficulty in reading an instrument or in manipulating the apparatus, then this should be taken into account when you estimate how good a measurement is.

(3) When possible, repeat the measurement. The average of many measurements is a better value to use for the measured quantity, and the variation in those measurements can give you a really good idea of how uncertain your measurement is. Formally, one could use the standard deviation as a measure of uncertainty, but in an elementary lab, we can get a good estimate of the uncertainty by taking 5 measurements, order them from highest to lowest and use half the difference between the 2^{nd} highest and 2^{nd} lowest.

If we measure our 5 pieces of paper as having lengths:

78.9mm
77.9mm
77.2mm
76.6mm
73.9mm

Then we would have an uncertainty due to variability in the results of (77.9mm-76.6mm)/2=0.65mm or keeping only one significant figure for uncertainty 0.7mm.

<u>Why You Might Still Be Wrong</u>

There are two other problems with experimental measurements that don't show up in uncertainty estimates. If these problems exist in your experiment, you can wind up with a measurement not agreeing with the expected result within uncertainty.

Maybe you screwed up. I'm not talking about being imperfect in positioning a meter stick here. I mean maybe you bumped the scale when it took a reading, misread the voltmeter, wrote down the wrong number, or performed a calculation incorrectly. If you catch it, you can fix the problem or repeat the experiment. If not, you wind up with a measured value that is really just wrong, and you don't know why.

As mentioned before, a more insidious problem is the <u>systematic error</u>. Maybe your clock runs slow, your scale isn't correctly zeroed, or there's friction that you haven't included in the analysis.

All of these problems would cause you to get experimental results that were systematically wrong. All your time measurements would run long or all you masses would be too big. Taking a bunch of measurement isn't going to help—their average will also be wrong! Again, if the systematic error goes unnoticed, the results of an experiment can disagree with the accepted or expected results, and you don't know why.

Propagation of Uncertainties

In general, you're not just going to measure the length of a piece of paper and compare it to the "expected" length of the paper. You're going to use that length in calculating some other quantity. How does the uncertainty in the paper's length translate to an uncertainty in that other quantity? Determining how the uncertainty of one measurement contributes to the uncertainty of another quantity is called propagation of uncertainties.

A useful intermediary between your measurement (with its absolute uncertainty) and the propagation of the error into the quantity you wish. Sometimes it is more useful to have a relative uncertainty. What percent of the measured value is the absolute uncertainty? So, for our example measuring the piece of paper, the relative uncertainty is

$$\frac{0.3\text{ mm}}{77.1\text{ mm}}\times 100\% = 0.4\%.$$

We could report the uncertainty of the measurement as

$$L = 77.1\text{ mm} \pm 0.4\%\,.$$

Suppose we have two measured quantities with their absolute uncertainties: $99 \pm 3\text{ m}$ and $21 \pm 1\text{ m}$. We could instead write these quantities to show their relative uncertainties: $99\text{ m} \pm 3\%$ and $21\text{ m} \pm 5\%$. There are many types of calculations we might want to do with one or both of those measurements. The most basic ones are described below.

- **Multiplication/division by a constant.**

Suppose we need to multiply the first measurement by two. When multiplying a quantity by a constant factor, its absolute uncertainty is also multiplied by that factor. Its relative uncertainty is unchanged.

In this example, the result of this calculation would be $198 \pm 6\text{ m}$ or $198\text{ m} \pm 3\%$.

- **Raising to a power or taking a root.**

Suppose we need to square the first measurement. When raising a quantity to some power, the easiest approach is to use the relative uncertainty. The relative uncertainty is multiplied by the power.

In this example, the power is two since we're squaring the measurement. So the result of this calculation would be $9800\ \text{m} \pm 6\%$.

- **Addition or subtraction of two measured quantities.**

Suppose we want to add or subtract the two measurements we have made. The easiest approach is to use the absolute uncertainty. In either case, the absolute uncertainty of the new result is the sum of the absolute uncertainties of the original measurements. Yes, even if you're subtracting the quantities.

In this example, adding our two measurements would lead to $120 \pm 4\ \text{m}$, and subtracting them would lead to $78 \pm 4\ \text{m}$.

If you are troubled with the fact that the uncertainties add when you're subtracting the quantities consider the following situation: start with two measurements with equal uncertainties subtracting those measurements cannot remove the uncertainty. You must add the uncertainties.

- **Multiplication or division of two measured quantities.**

Suppose we want to multiply or divide the two measurements we have made. The easiest approach is to use the relative uncertainty. In either case, the relative uncertainty of the new result is the sum of the relative uncertainties of the original measurements.

In this example, multiplying our two measurements would lead to $2080\ \text{m}^2 \pm 8\%$, and dividing them would lead to $4.7 \pm 8\%$.

- **Realistic example of propagating uncertainties.**

Suppose you are determining the change in kinetic energy of a moving object (due to the action of a force acting on it.) You measure the object to have a mass of $m = 0.158 \pm 0.003\ \text{kg}$. It begins with a velocity of $v_i = 5.9 \pm 0.2\ \text{m/s}$ and ends with a velocity of $v_f = 2.4 \pm 0.1\ \text{m/s}$. By how much has its kinetic energy changed, and what is the uncertainty in that result?

The first part of the question is easy. Since the kinetic energy of a moving object is given by $K = \frac{1}{2}mv^2$, the change in kinetic energy is found from $\Delta K = \frac{1}{2}m\left(v_f^2 - v_i^2\right) = -2.3\ \text{J}$. But what is the uncertainty?

The calculation process has involved squaring velocities, subtracting those squared velocities, multiplying by the mass, and multiplying by a constant. Since we must first square the velocities, we'll need relative uncertainties to make that job easier. Writing the velocities with their relative uncertainties yields

$$v_i = 5.9\,\text{m/s} \pm 3.4\% \quad \text{and} \quad v_f = 2.4\,\text{m/s} \pm 4.2\% \ .$$

Squaring a quantity means we double its relative uncertainty, so the velocities squared are written

$$v_i^2 = 34.81\,\mathrm{m^2/s^2} \pm 6.8\% \quad \text{and} \quad v_f^2 = 5.76\,\mathrm{m^2/s^2} \pm 8.4\% \quad .$$

We now want to subtract these two results, so we'll need to know their absolute uncertainties to do that.

$$v_i^2 = 34.81 \pm 2.37 \ \ \mathrm{m^2/s^2} \ \ \text{and} \ \ v_f^2 = 5.76 \pm 0.48 \ \ \mathrm{m^2/s^2} \ .$$

Subtracting these results means we should add their absolute uncertainties.

$$v_f^2 - v_i^2 = -29.05 \pm 2.85 \ \ \mathrm{m^2/s^2} \quad .$$

To multiply by $\frac{1}{2}m$, it would be easiest to switch back to relative uncertainties:

$$v_f^2 - v_i^2 = -29.05\,\mathrm{m^2/s^2} \pm 9.8\% \ \ \text{and} \ \ m = 0.158 \ \mathrm{kg} \pm 1.9\% \quad .$$

The final result is then $\Delta K = \frac{1}{2}m\left(v_f^2 - v_i^2\right) = -2.3 \ \mathrm{J} \pm 12\%.$ Switching back to absolute uncertainties would give us $\Delta K = -2.3 \pm 0.3 \ \mathrm{J}$.

In reporting final results like these, I've used an appropriate number of digits in writing the uncertainty. It wouldn't make sense to use $\pm 11.7\%$ or ± 0.269 J. After all, is our uncertainty that certain? No way.