## 1 Valid Argument Forms by Truth Tables

There are more valid argument forms than just the basic valid argument forms that we saw in Lecture 6. If an argument form has no quantifiers, writing down a truth table is always one option of determining whether the argument form is valid.

As an example, show that this argument form is valid.

$p$
$r$
$\therefore \neg p \implies r$

To make the truth table more manageable, let $A = p \wedge r$ denote the conjunction of the assumptions and let $C = \neg p \implies r$ denote the conclusion. The goal is to show that $A \implies C$ is a tautology.

| $p$ | $r$ | $A = p \wedge r$ | $\neg p$ | $C = \neg p \implies r$ | $A \implies C$ |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | F | F | T | T |
| F | T | F | T | T | T |
| F | F | F | T | F | T |

## 2 New Valid Argument Forms from Known Valid Argument Forms

Sometimes known valid argument forms can be used to show that another argument form is valid. As an example, consider the following argument:

$2 < 3 \implies 1 < 3$
$1 < 3 \implies 0 < 3$
$2 < 3$
$\therefore, 0 < 3$

This has the following argument form:

$p \implies q$
$q \implies r$
$p$
$\therefore r$

Now suppose we want to show that this is a valid argument form. Unfortunately, it is not one of our basic valid argument forms. We could accomplish this with a truth table. Since there are three variables, the truth table will have 8 lines for all the possible combinations of truth values of $p$, $q$ and $r$.

Here there is a simpler way to proceed. From the first two lines, we can conclude $p \implies r$ by transitivity of $\implies$. Using this and the third line, we can conclude $r$ by direct implication. Here is a nice way of organising this reasoning.

| | | |
|---|---|---|
| (1) | $p \implies q$ | given |
| (2) | $q \implies r$ | given |
| (3) | $p \implies r$ | transitivity of $\implies$, lines (1) and (2) |
| (4) | $p$ | given |
| (5) | $r$ | direct implication, lines (3) and (4) |

We list the statements we know to be true. Some statements are true because they are premises in the argument. Other statements are true because they follow from known statements and the use of known basic valid argument forms. We use line numbers to demonstrate exactly how we are obtaining each result. In the very last line, we obtain our conclusion. This demonstrates that our argument form is valid.

## 3 Showing an Argument is Not Valid

An argument is valid when its argument form is valid. As an example, show that the argument from Homework 3, Problem 9 is not valid.

$\forall x \in \mathbb{R},\, x < |x| \implies x < 0$
$-3 \in \mathbb{R}$
$-3 < 0$
$\therefore -3 < |-3|$

Notice that the premises and conclusion of this argument are all true. The first step is to find the argument form.

$\forall x \in U,\, p(x) \implies q(x)$
$a \in U$
$q(a)$
$\therefore p(a)$

Now we find an example of an argument which has this form and whose premises are true and conclusion is false. To do this, we need to understand the structure of this argument form. In the first line, a certain implication is assumed to be universally true. In the second line, a specific element is considered. By specification, the implication on the first line is true for this element. That is, we could conclude from the first two lines that $p(a) \implies q(a)$. In the third line, $q(a)$ is assumed, and on the fourth line, it is concluded that $p(a)$ is true. We need to come up with an example where $q(a)$ is true but $p(a)$ is false.

$\forall x \in \mathbb{R},\, x > 0 \implies x^2 > 0$
$-6 \in \mathbb{R}$
$(-6)^2 > 0$
$\therefore -6 > 0.$

Here, the premises are all true but the conclusion is false. Therefore this argument is invalid. The trick was to pick an implication that only goes one way. Here, $x > 0$ implies $x^2 > 0$, but not vice versa. In the original example, $x < |x|$ if and only if $x < 0$.

## 4 Proving that an Argument Form with Universals is Valid

To prove that an argument form is valid, we need to demonstrate that the conclusion follows from the premises.

$\forall x \in U, p(x) \wedge \neg q(x)$
$\forall x \in U, q(x) \vee r(x)$
$\therefore, \forall x \in U, \neg p(x) \implies r(x)$

We do this by writing down a separate list of known facts. We can use the given information to obtain intermediate results, and put these together to obtain the conclusion. To prove a universal statement, we must use the generalisation principle. Therefore, we start out by choosing an arbitrary element of $U$.

| | | |
|---|---|---|
| (1) | Let $a \in U$ be arbitrary | |
| (2) | $\forall x \in U, p(x) \wedge \neg q(x)$ | given |
| (3) | $p(a) \wedge \neg q(a)$ | specification, lines (1) and (2) |
| (4) | $\forall x \in U, q(x) \vee r(x)$ | given |
| (5) | $q(a) \vee r(a)$ | specification, lines (1) and (4) |
| (6) | $p(a)$ | in particular, line (3) |
| (7) | $\neg q(a)$ | in particular, line (3) |
| (8) | $r(a)$ | eliminating possibility, lines (5) and (7) |
| (9) | $\neg p(a) \implies r(a)$ | see earlier in this lecture, lines (7) and (8) |
| (10) | $\forall x \in U,\, \neg p(a) \implies r(a)$ | generalisation, lines (1) and (9) |

In line (10), we have obtained the conclusion. Thus, this argument form is valid.